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A theoretical study of automatic inertial navigation

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Cambridge, Massachusetts; Massachusetts Institute of Technology

A THEORETICAL STUDY OF AUTOMATIC INERTIAL NAVIGATION

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A THEORETICAL STUDY OF AUTOMATIC INERTIAL NAVIGATION

by

Leonard Erb — B.S., United States Naval Academy, 1941

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
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May 20, 1949

Prof. Joseph S. Newell
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

Dear Professor Newell:

In accordance with the regulations of the faculty, we hereby submit a thesis entitled, A THEORETICAL STUDY OF AUTOMATIC INERTIAL NAVIGATION in partial fulfillment of the requirements for the degree of Master of Science.

ACKNOWLEDGMENT

The authors wish to express their deep appreciation to Dr. C. S. Draper and to Dr. W. Wrigley for the interest, confidence, and assistance given by them in supervising the progress of this work.

Invaluable aid was also given by the staff of the Instrumentation Laboratory, particularly of the "Febe" Section. Individual mention is foregone only because of the many members who helped.

The staff of the Rockefeller Analyzer, at the Massachusetts Institute of Technology, by their interest and cooperation, contributed greatly to the work.

Thanks are given to Mr. L. E. Payne and his associates of Jackson & Moreland for their part in the preparation of the text and illustrations.

ABSTRACT

The purpose of this thesis was to postulate a method of long-range inertial navigation, using only acceleration inputs, and to examine theoretically the response of this system to external disturbances, using various types of mechanizations. The problem was divided into one of track control and of range indication, using a great-circular path between the points of departure and destination.

The response of the track control system was examined with mechanization equations of as high an order as the fifth, and was found to improve as the order of the mechanization equation increased. Response to impulse type wind acceleration disturbances was satisfactory, but long-period sinusoidally varying winds caused excessive errors. It appears probable that additional feedback loops, in conjunction with a mechanization equation of the fifth or higher order, will solve the track control problem satisfactorily.

In the range indication system, it was found to be impossible to remove any forcing function terms of higher order than the time rate of change of acceleration. It appears probable that the range indication system, as postulated, will prove to be satisfactory, especially if the sensitivities are made variable, as a function of the angle of the input pendulum.

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CHAPTER I

INTRODUCTION

The importance of developing a long-range self-contained system of automatic, mid-course, navigation for the control of guided missiles cannot be overestimated (Fig I-1). As long range flights of piloted aircraft over enemy territory become increasingly hazardous, because of the necessity of travelling without fighter protection toward heavily defended cities provided with elaborate warning nets, automatic navigation can provide a method of delivering bombs without risking the lives of highly trained pilots.

With the development of long-range jet-propelled missiles, automatic navigation should eventually permit the United States to bomb any place in the world from bases located within its territorial limits.

For such a system of automatic navigation to be successful, it must have a degree of accuracy comparable to that achieved by human pilots. It must not be easily susceptible to "jamming" by enemy action. It must operate successfully at supersonic velocities, and in the upper reaches of the earth's atmosphere.

At present, the problems of automatic long-range navigation are under consideration by several groups. Among these are:

1. Baird Associates, Inc.
2. Hughes Aircraft Co.
3. Instrumentation Laboratory, M.I.T.
4. Kollsman Instrument Company
5. North American Aviation, Inc.
6. Northrup Aircraft, Inc.

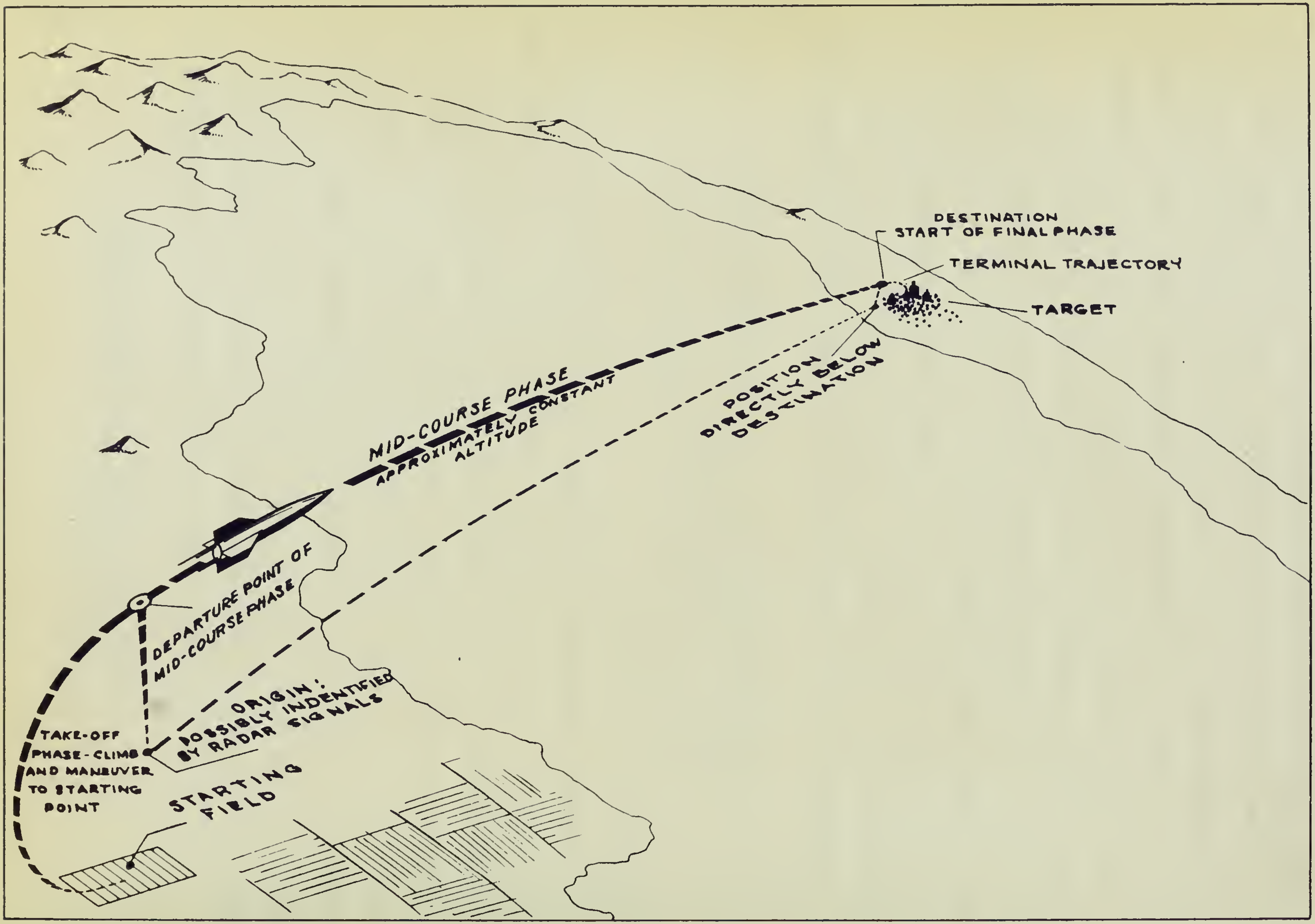

FIG. I-1 RELATIONSHIP BETWEEN PHASES OF A LONG RANGE MISSILE TRAJECTORY.

7. Rand Corporation
8. U. S. Army Research and Development Sub-office (rocket), Fort Bliss, Texas

Little work, however, appears to have been done on the derivation of theoretically optimum control equations or operating parameters. For this reason the writers determined to examine a long-range automatic navigation system now under development at the Instrumentation Laboratory, M.I.T., as a means for exploring theoretically some of the general problems of automatic navigation. The Instrumentation Laboratory project, which is being executed under USAF Contract W33-038ac-13969, is designated "An Automatic Navigation System - Project Febe," and will be the subject of a report soon to be published. This project has for its primary purposes to determine:

1. the feasibility of long-range automatic navigational guidance of bomber airplanes
2. useful design parameters for a serviceable system for military use.

Considerations other than military and tactical indicated that the system should employ solar tracking, a magnetic azimuth system, and a constant ground speed. The system is allowed to make no contact with the earth. Altitude is determined by the use of a barometric altimeter; the only inputs to the system are celestial observation, the magnetic field of the earth, and the various accelerations experienced by the airplane.

This system, which has been installed in an Air Force B-29 bomber, causes the airplane to fly a definitely programmed great circle course at constant ground speed. Vertical accelerations are assumed to be of such small importance that they can be neglected. Figure I-2 shows a functional diagram of the Febe automatic navigational system. During the period from

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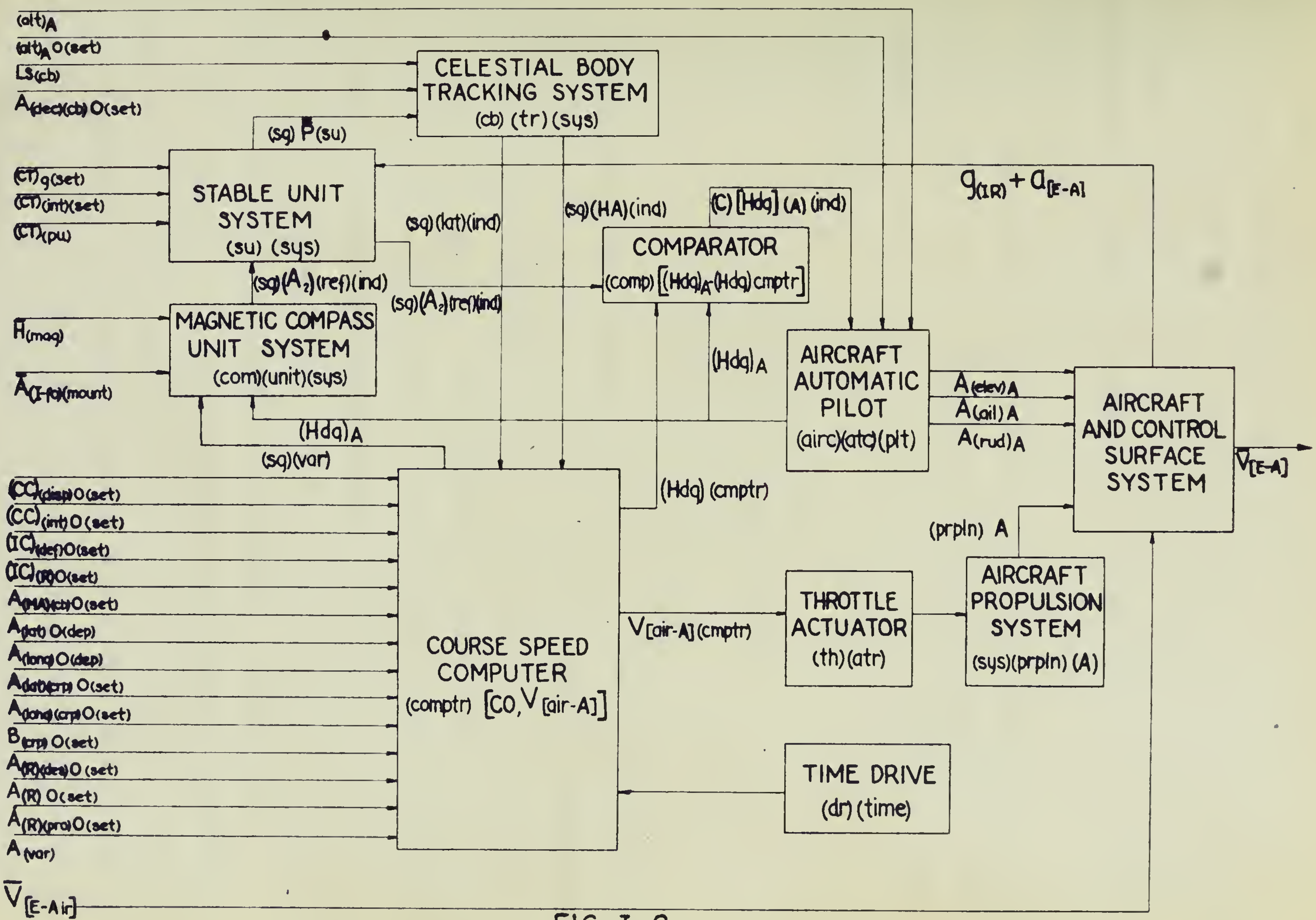

FIG. I-2

FUNCTIONAL DIAGRAM OF FEBE SYSTEM FOR AUTOMATIC NAVIGATION

June to August of 1948, the writers of this thesis were concerned with the Febe project in an under-instruction status. Through the courtesy of Dr. C. S. Draper, they were also permitted to attend a Seminar on Automatic (Celestial and Inertial) Long-Range Guidance Systems conducted at the Massachusetts Institute of Technology by the Scientific Advisory Board from February 1 to February 3, 1949.

This background naturally led the writers to concern themselves with the form which a practicable service guidance system might take. The Febe system presents several technical and military problems, which arise principally in connection with the following:

1. Constant ground speed
2. The necessity of solar tracking
3. Magnetic azimuth input
4. Weight and size of components.

Since a successful automatic navigational system is ultimately destined to fly in a long-range high-speed missile, it must be assumed that the weapon will be operating at close to the maximum range permitted by its size and fuel capacity. The navigational system, therefore, should not seriously lower the fuel economy. This suggests a system which will fly at constant, or nearly constant, airspeed. Furthermore, if the power plant consists of an athodyd, the airspeed must be maintained very nearly constant by the fundamental limitations of this type of propulsion.

The limitation of the Febe system to a constant ground speed was largely dictated by the requirements of the azimuth system. The magnetic input could be eliminated through the use of two star-tracking telescopes. The problems of celestial tracking become increasingly severe, however, as missile speed is increased. At supersonic and near-sonic speeds, it is

no longer feasible to have an astrodome, in which to house the tracking unit, projecting from the fuselage. Thus, tracking through a flat window, with the attendant difficulties which this entails, becomes a necessity. Also, at high speeds, thick boundary layers and intense heating will exist along the surface of the missile, greatly complicating optical problems. Finally, the inclusion of a celestial tracking system materially increases the space and weight required by the navigational system — and weight and space factors become increasingly important with increasing range and bomb load of an aircraft. It appears, however, that approximately a thirty-fold decrease in the uncertainty levels of existing gyros would permit the maintenance of an inertial coordinate reference system within the missile through the use of such gyros alone, thus eliminating the use of celestial tracking.

Great progress is being made in the improvement of gyros. The work, for example, that is being done by the group at the Massachusetts Institute of Technology under Dr. C. S. Draper shows promise of obtaining the necessary accuracy within the next few years.

The model of the Febe automatic navigational system presently installed in a B-29 is very bulky, and weights 1917 lbs. (See Fig 1-3). It practically fills the after pressurized compartment of the airplane. Of course, it must be remembered that no effort was made to decrease the size of this equipment, or to keep its weight at a minimum. Nevertheless, a missile held to a constant airspeed, rather than to a programmed ground spped, could, by using gimbal solvers, greatly simplify the elaborate trigonometric and speed computers which Febe requires. Much of the work of the computation systems might, of course, be performed on the ground

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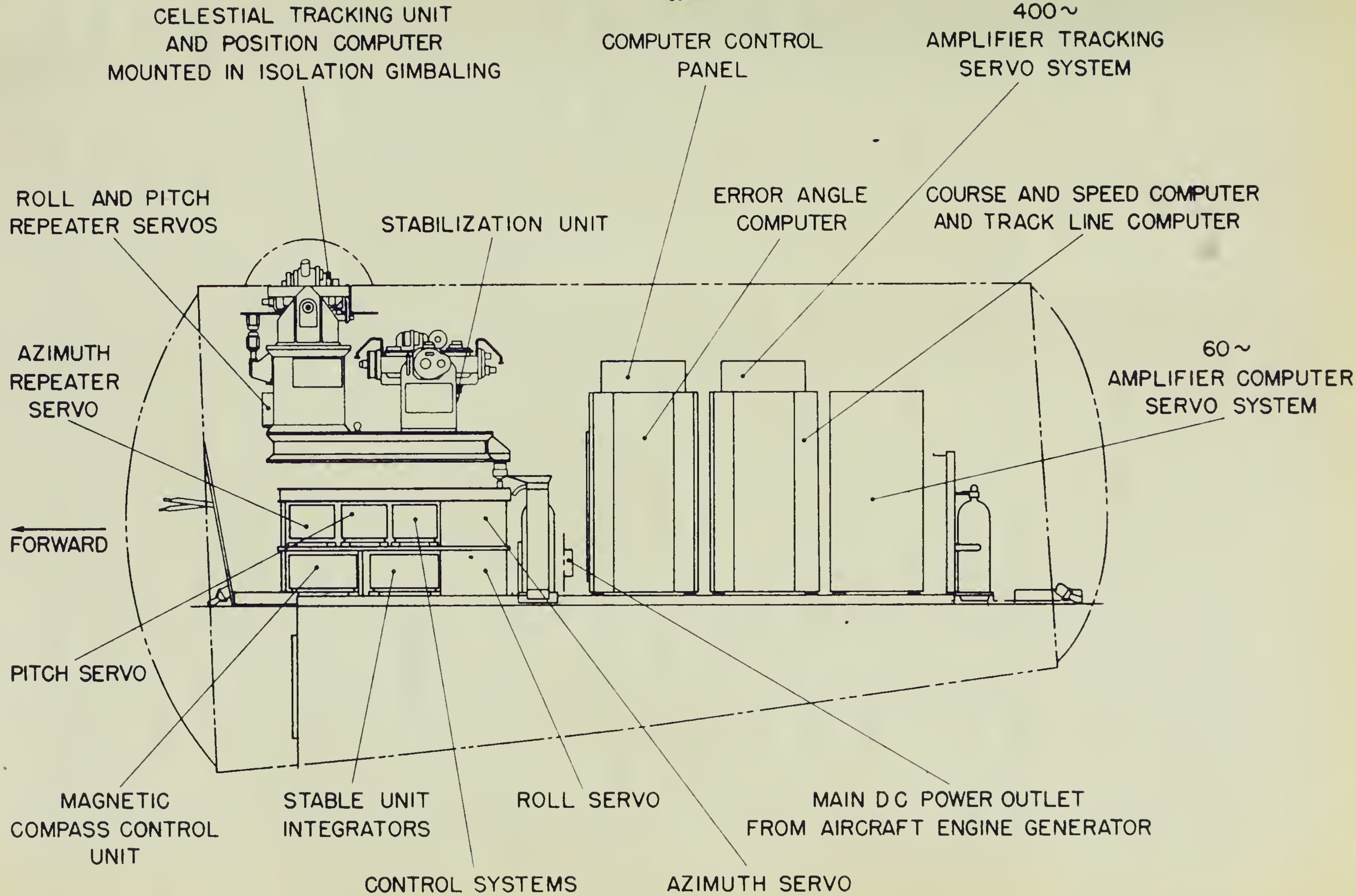

FIG. I-3

ELEVATION: B-29 AFT PRESSURIZED COMPARTMENT INSTALLATION OF FEBE AUTOMATIC NAVIGATION SYSTEM.

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prior to the start of the flight, and be fed to the system through tapes, but this would introduce further complications.

From all of these considerations, a self-contained inertial-gravitational system flying a great circle course at nearly constant airspeed appears to provide one of the most satisfactory solutions, if such a system can be implemented.

After this examination of the Febe system, there remain, then, for use in the purely inertial system, a gyro-stabilized inertial platform, a clock mechanism to remove the earth's diurnal rotation, and two mutually perpendicular single-degree-of-freedom pendulous accelerometer units. In addition, there are gimbals and servomechanisms isolating the inertial platform from the motions of the missile containing it, and orienting the platform so that its axis is parallel to the polar axis of the earth, so that a reference vertical parallel to that of the point of departure (or any other convenient reference direction) is maintained independent of the motion of the missile. (See Figs I-4, I-5 and I-6.)

The three integrating gyros which are used to maintain a gimbal-mounted inertial platform "fixed" in inertial space, parallel to the earth's equatorial plane, are mounted so that they detect motion of the platform with respect to inertial space about the polar axis of the earth, and about two mutually perpendicular axes parallel to the plane of the equator (Figs. I-7, I-8). Servomechanisms, controlled by these gyros, rotate the gimbals so that the inertial platform remains fixed in inertial space. The earth platform, mounted on the inertial platform with the identical polar axis, is rotated about that axis once in twenty-four hours, to transform the inertial reference system of the inertial platform into an earth reference system.

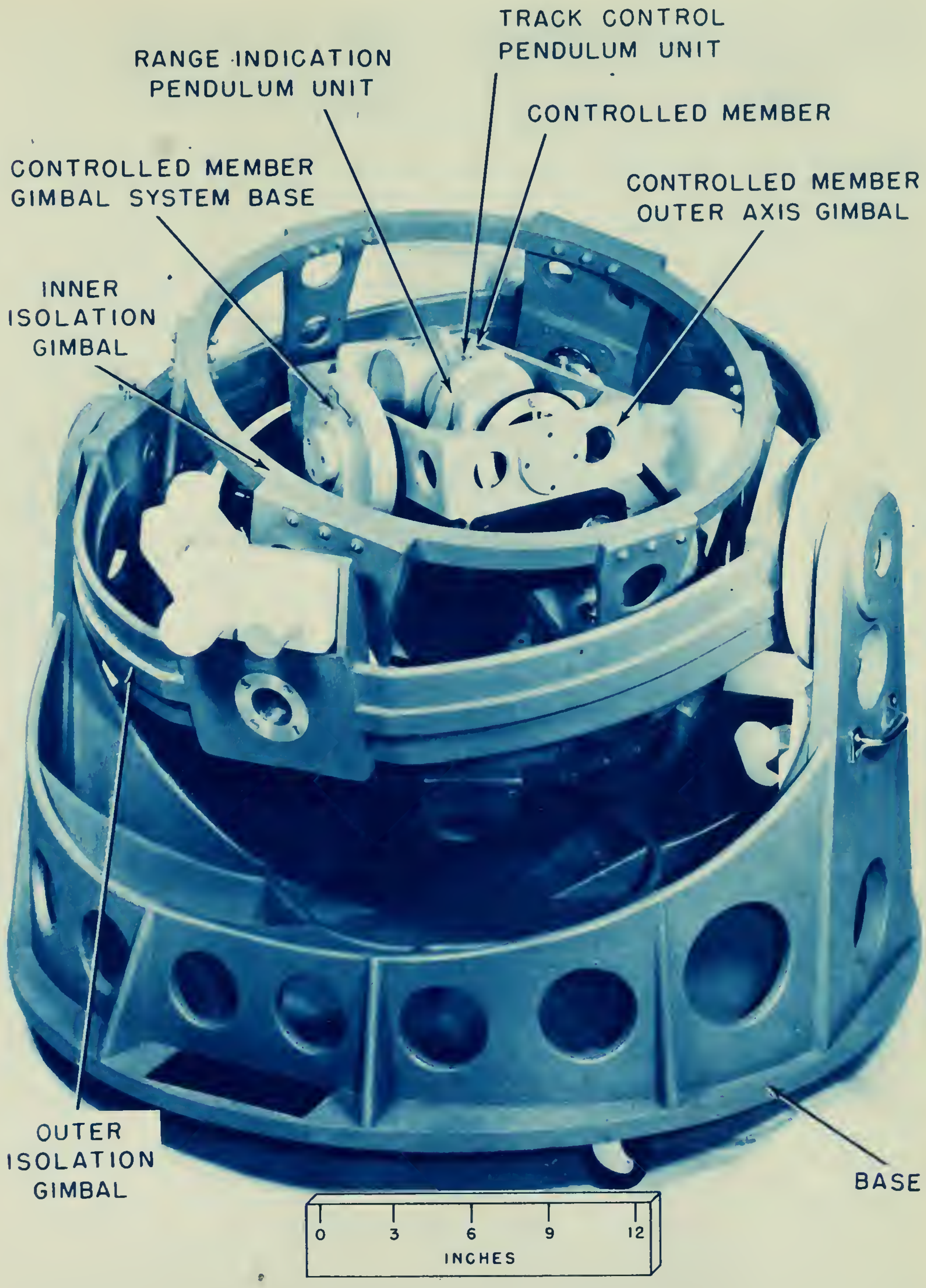


FIG. I-4.
PHOTOGRAPH OF INERTIAL REFERENCE AUTOMATIC NAVIGATION SYSTEM MODEL.

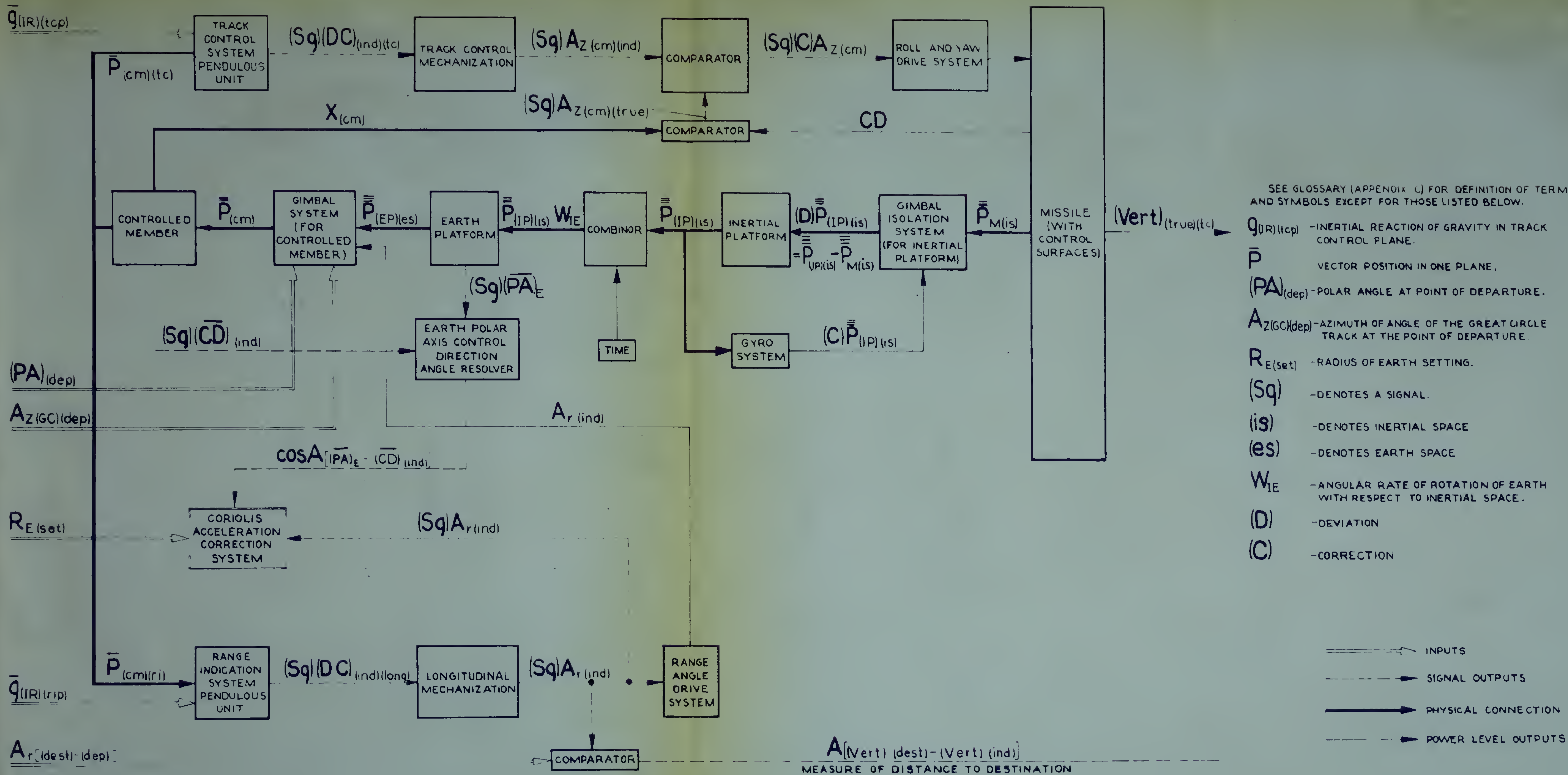


FIG. I-5. SIMPLIFIED FUNCTIONAL DIAGRAM OF AUTOMATIC INERTIAL GUIDANCE SYSTEM.

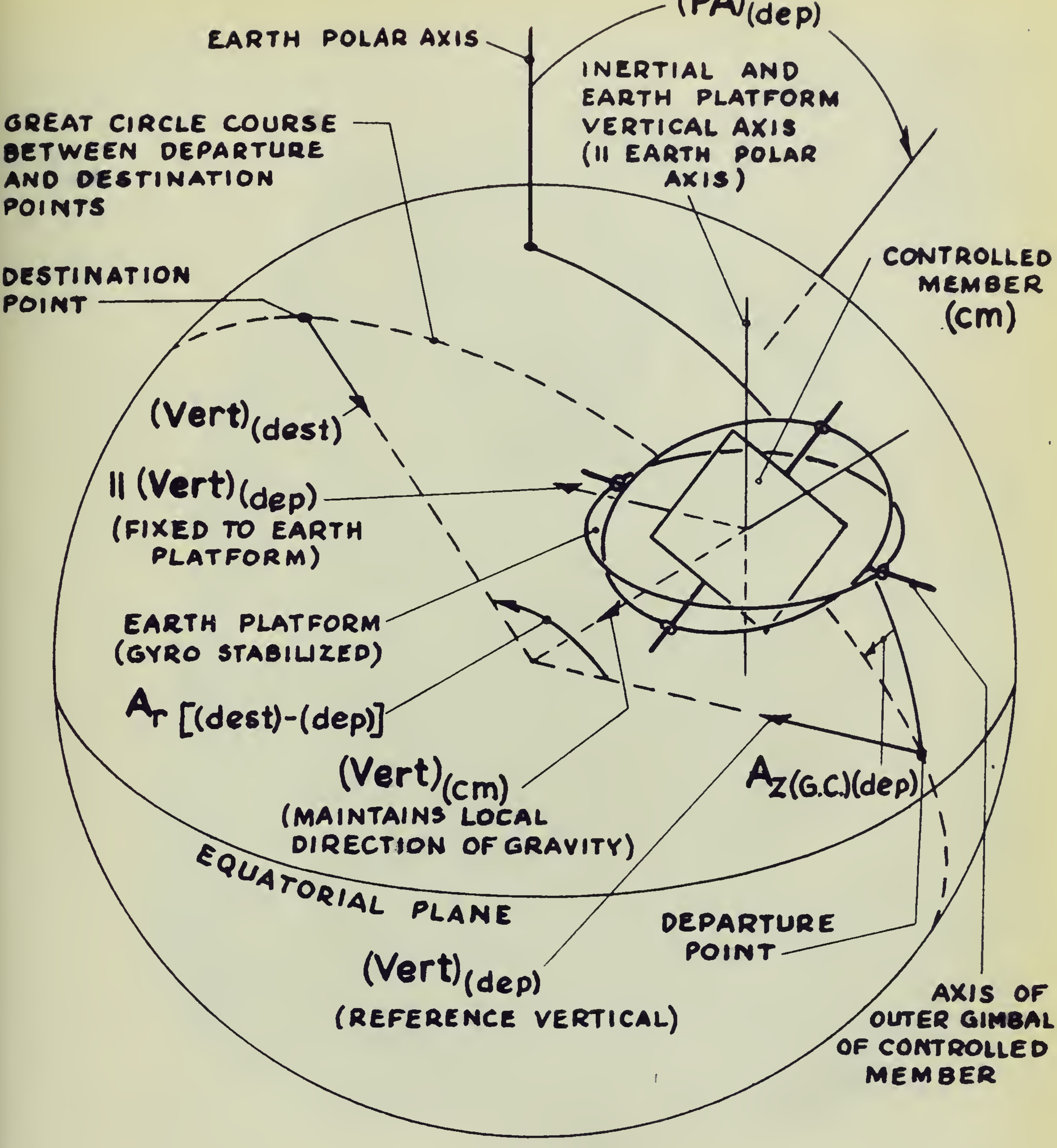


FIGURE I-6 GEOMETRIC DIAGRAM OF AXES OF AUTOMATIC NAVIGATION SYSTEM

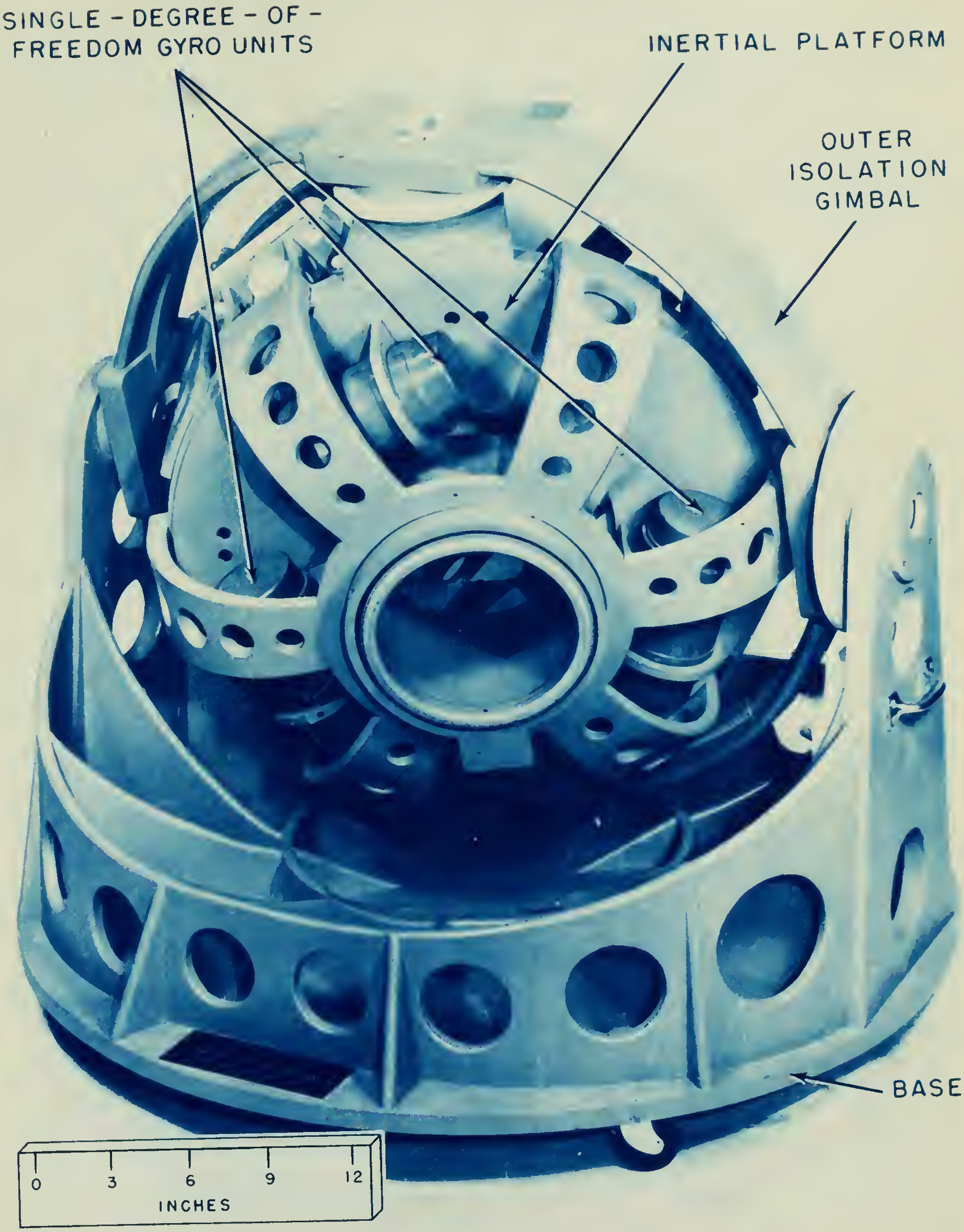


FIG. I-7.
PHOTOGRAPHIC VIEW OF INERTIAL REFERENCE AUTOMATIC NAVIGATION SYSTEM MODEL WITH POLAR AXIS TIPPED AWAY FROM THE NORMAL DIRECTION TO THE BASE.

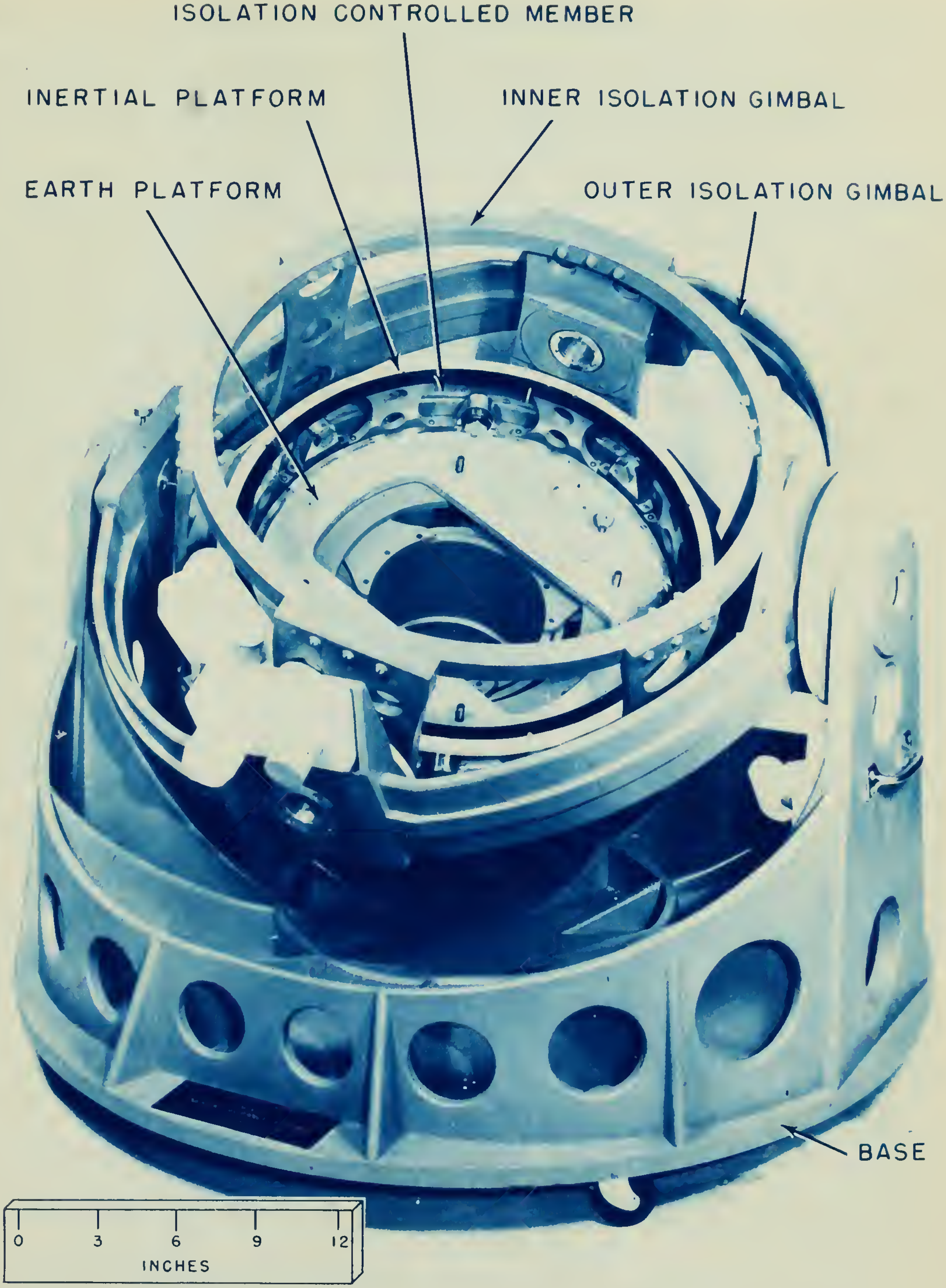


FIG. I-8.
PHOTOGRAPHIC TOP VIEW OF INERTIAL REFERENCE AUTOMATIC NAVIGATION SYSTEM MODEL WITH CONTROLLED MEMBER GIMBAL SYSTEM REMOVED.

Upon the earth platform, in turn, the controlled member is mounted in gimbals (Fig I-9). The axis of the outer gimbal of this member is made to lie in the plane of the great circle connecting the points of departure and destination on the surface of the earth, considered as a sphere.* Then, about the inner controlled-member gimbal axis, the controlled-member is moved until it lies parallel to the surface of the earth at the point of departure. The inner controlled-member axis is placed perpendicular to the programmed great circle path so that, as the missile moves along the programmed path, the controlled member, by rotation about the inner axis alone, can be maintained at all times parallel to the surface of the earth below the missile. The angle through which the controlled member has rotated will then be a direct reading of the distance travelled by the missile along the programmed great circle. At some predetermined angle between the controlled member and the gimbal connecting it to the earth platform, the missile may be said to have arrived at its destination.

The two pendulous units are mounted upon the controlled member, one with its plane of motion in that of the programmed great circle, the other with its plane of motion perpendicular to that great circle plane. The longitudinal pendulum, i.e. the unit with its input plane parallel to the plane of the programmed track, sensing longitudinal accelerations, produces an output signal which is then modified through the implementation of the longitudinal mechanization equation. The resultant signal is used as the input

* This orientation is accomplished by rotating the controlled member on the earth platform to the correct meridian angle of the great circle track at the point of departure, $[A_{Z(G.C.)(dep)}]$, and by tilting the outer gimbal axis to the correct polar angle $(PA)_{(dep)}$ of the great circle track plane with respect to the polar axis of the earth.

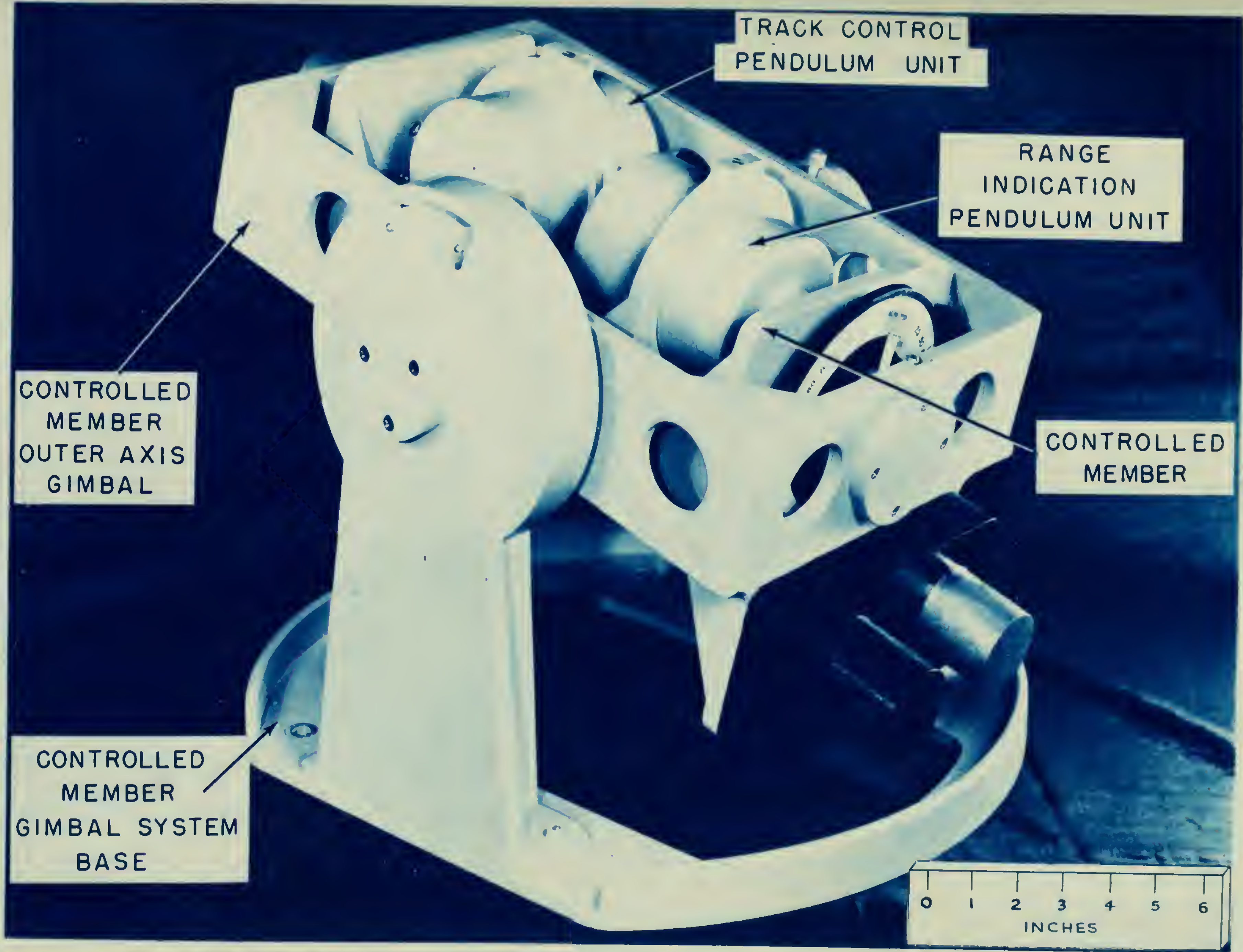


FIG. I-9

PHOTOGRAPH OF CONTROLLED MEMBER MODEL WITH PENDULOUS UNITS IN PLACE

to a servodrive system that rotates the controlled member so that it tracks the local vertical. Similarly, the track control pendulum, sensing lateral accelerations; (i.e. accelerations perpendicular to the great circle track plane), produces an output which is modified through the implementation of the lateral mechanization equation, and the resultant is used to position the missile control surfaces so that the missile remains close to the programmed great circle track. The altitude of the missile above the surface of the earth is maintained by a pressure type altimeter that supplies the essential input to the altitude control system.

A Coriolis acceleration computer is used to compensate for the effects of the Coriolis acceleration upon the action of the system. Geodesic acceleration is in general small enough to be ignored. It will be the purpose of this thesis to discover whether it is theoretically possible to maintain the errors between the indicated and actual local verticals small enough to permit sufficiently accurate indication of the local vertical for the long-range bombing and missile guidance problems. Various mechanization equations will be examined, to see which offer the greatest chances of success. The most promising of these will be furthur examined to deduce the optimum control parameters.

CHAPTER II

KINEMATICS

1. Purpose

It is the purpose of this chapter to derive the fundamental kinematic equations describing the motions of the missile postulated in Chapter I. The principle geometrical relationships are illustrated in Fig II-1, and are shown symbolically in Fig II-2.

The following simplifying assumptions are made in deriving the kinematic equations:

a. Lateral and longitudinal motions decoupled
b. Constant altitude flight path
c. Spherical and homogeneous earth
d. Controlled member located at center of gravity of missile
e. Zero angle of attack and side slip angle

The decoupling of the lateral and longitudinal motions divides the problem into one of control of the missile in the direction perpendicular to the programmed great circle and one of indication of the distance travelled in the programmed great circle. The kinematic equations are therefore derived separately for lateral and longitudinal motions.

Since the flight path is to be considered as having constant altitude above the surface of the earth, the only vertical component of acceleration is the inertia reaction acceleration of gravity. With the earth assumed to be spherical and homogeneous, this component is a constant over the surface of the earth, with the numerical value of 115,920 feet per minute per minute (equivalent to 32.2 ft/sec^2). The assumption of a spherical earth also eliminates any geodesic acceleration terms from the kinematic equations (ref par 1 App A).

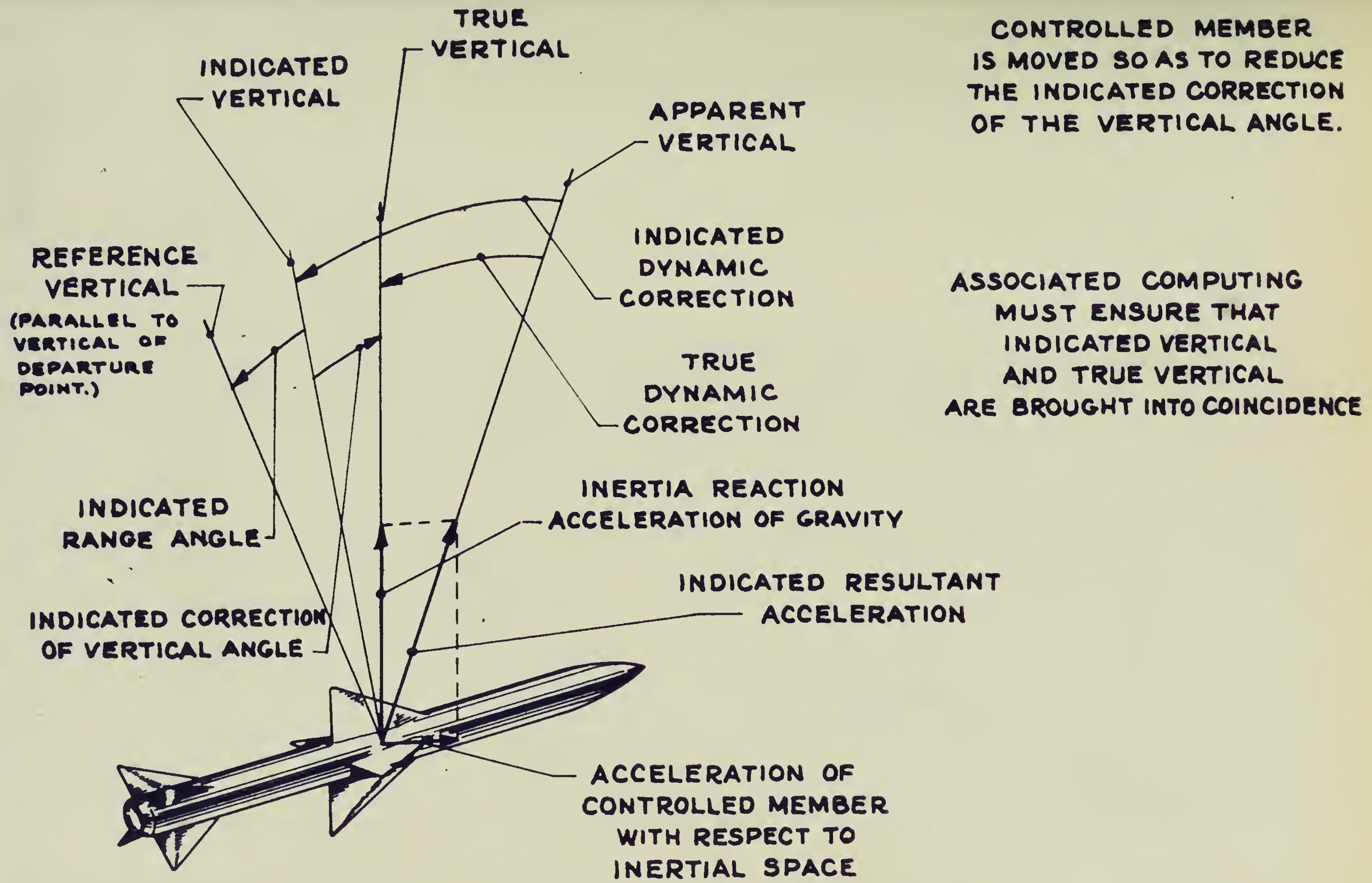
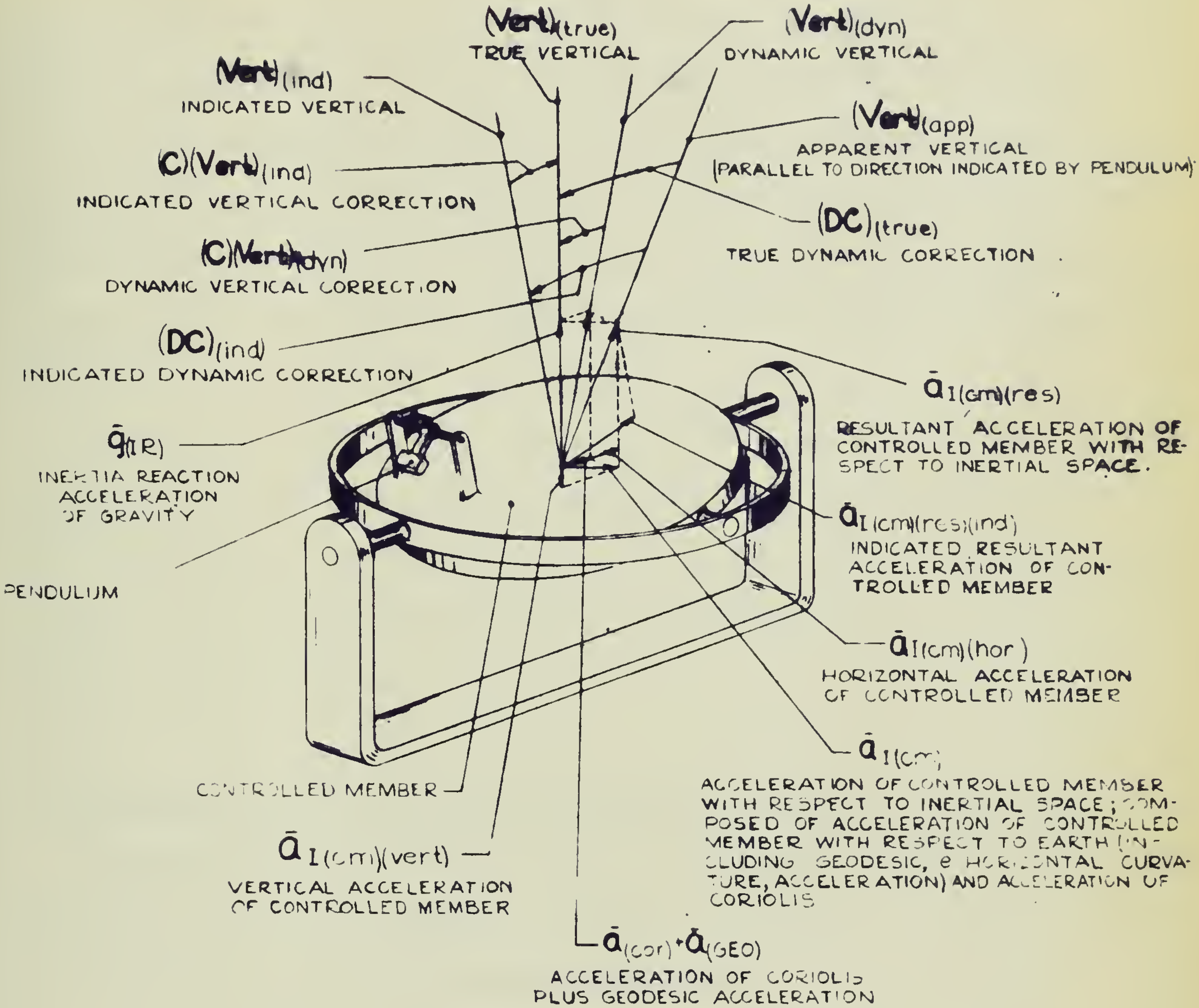


FIG. II-1. GEOMETRICAL RELATIONSHIPS AMONG PHYSICAL QUANTITIES ASSOCIATED WITH TRACKING IN AN INERTIAL GUIDANCE SYSTEM.



TRUE VERTICAL - DIRECTION OF INERTIA REACTION ACCELERATION OF GRAVITY AT CONTROLLED MEMBER.

DYNAMIC VERTICAL - DIRECTION OF VECTOR RESULTANT OF INERTIA REACTION ACCELERATION OF GRAVITY, ACCELERATION OF CORIOLIS AND GEODESIC ACCELERATION AT CONTROLLED MEMBER.

APPARENT VERTICAL - DIRECTION OF RESULTANT ACCELERATION OF CONTROLLED MEMBER WITH RESPECT TO INERTIAL SPACE.

INDICATED VERTICAL - DIRECTION FIXED TO CONTROLLED MEMBER - PARALLEL TO DIRECTION OF PENDULOUS ELEMENT WHEN ITS OUTPUT SIGNAL IS ZERO WITH CONTROLLED MEMBER STATIONARY.

FIG. II-2.

GEOMETRICAL RELATIONSHIPS AMONG DIRECTIONS ASSOCIATED WITH THE DIFFERENCES BETWEEN THE INERTIA REACTION ACCELERATION OF GRAVITY AND THE DIRECTION OF ACCELERATIONS ON A MOVING BASE.

A Coriolis acceleration computer is included in the fundamental guidance system (Fig I-5), and is assumed to work ideally (ref par 2 App A). The kinematic equations, therefore, do not contain the Coriolis acceleration.

With the controlled member located at the center of gravity of the missile, the gimbal isolation system can be considered to remove entirely all effects of the roll and pitch of the missile. No accelerations reach the controlled member as a result of angular accelerations of the missile about its body axes.

Considering both the angle of attack and the angle of side slip as zero, the velocity vector of the missile always lies along the longitudinal body axis of the missile; that is, $V_{[(air) - M]}$ lies along X_M (see Fig II-3). This is taken as the control direction (CD). The kinematic equations for lateral motion are derived first.

2. Kinematic Equations of Motion Perpendicular to Great Circle Track*

The geometrical elements of the simplified lateral guidance problem are illustrated in Fig II-4. Figure II-5 defines the directions and angles that are used in the derivation of the kinematic equations.

As can be seen from Fig II-5, the following relations are true:

$$(C)A_{(Vert)(tc)} = (DC)_{(true)(tc)} - (DC)_{(ind)(tc)} \tag{II-1}$$

$$a_{[E-(cm)](tc)} = R_E \, (C)\ddot{A}_{(Vert)(tc)} \tag{II-2}$$

$$\tan(DC)_{(true)(tc)} = \frac{-a_{[E-(cm)](tc)}}{g_{IR}} \tag{II-3}$$

*This development follows that of John Hutzenlaub, Notebook, July, 1947.

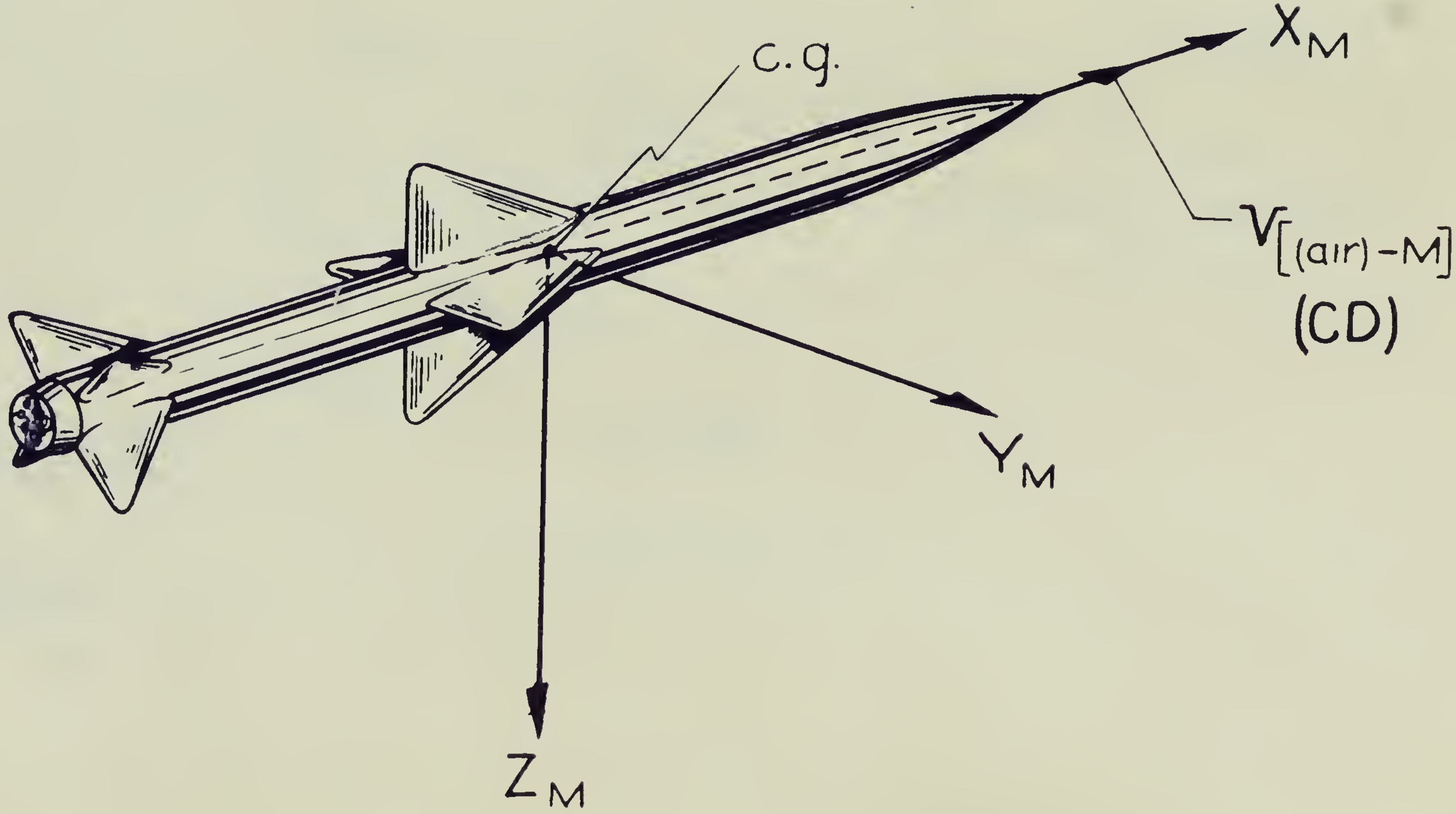


FIG. II-3 MISSILE COORDINATES

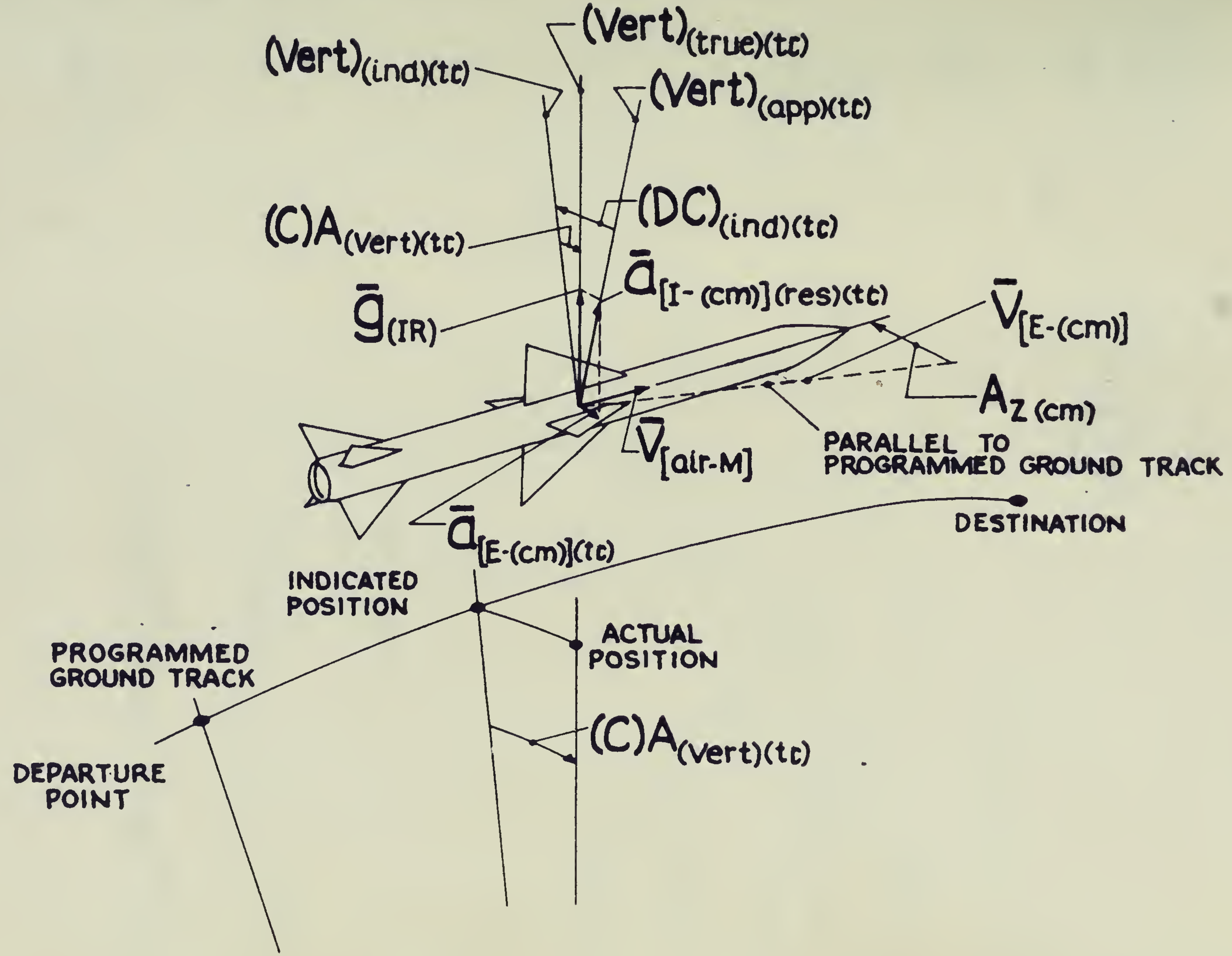


FIG. II-4 GEOMETRICAL ELEMENTS OF TRACK CONTROL PROBLEM.

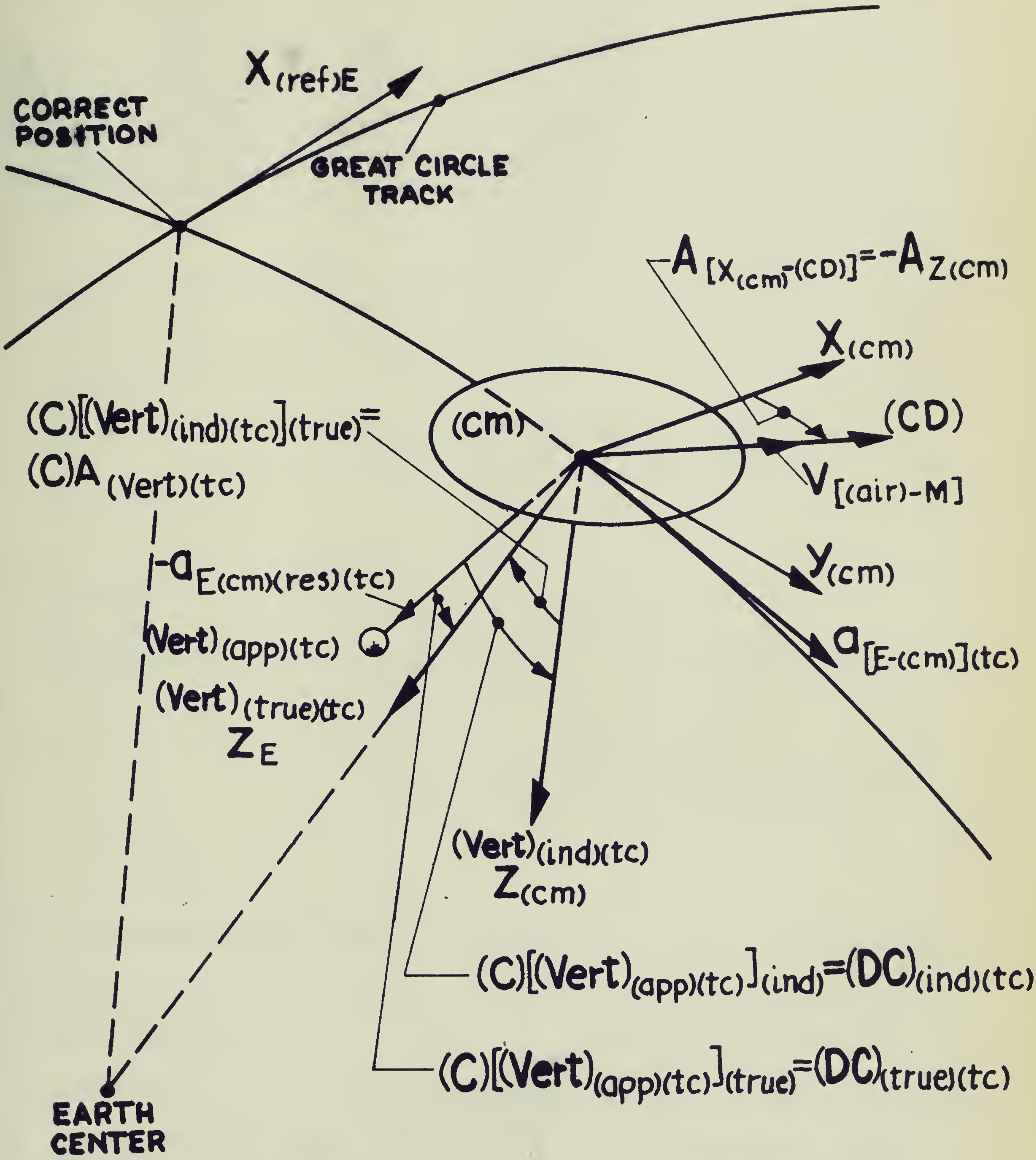


FIGURE II-5. TRACK CONTROL DIAGRAM

$$V_{[E-(cm)](tc)} = V_{[E-(air)](tc)} - V_{[(air)-M]} \sin A_{Z(cm)} \cos (C)A_{(Vert)(tc)} \tag{II-4}$$

From (II-2) and (II-3)

$$R_E (C)\ddot{A}_{(Vert)(tc)} = -g_{IR} \tan (DC)_{(true)(tc)} \tag{II-5}$$

But, from (II-1)

$$\tan (DC)_{(true)(tc)} = \tan \left[(C)A_{(Vert)(tc)} + (DC)_{(ind)(tc)}\right]$$

$$= \frac{\tan (C)A_{(Vert)(tc)} + \tan (DC)_{(ind)(tc)}}{1 - \tan (C)A_{(Vert)(tc)} \tan (DC)_{(ind)(tc)}} \tag{II-6}$$

Therefore,

$$-\frac{R_E}{g_{IR}} (C)\ddot{A}_{(Vert)(tc)} = \frac{\tan (C)A_{(Vert)(tc)}}{1 - \tan (C)A_{(Vert)(tc)} \tan (DC)_{(ind)(tc)}} + \frac{\tan (DC)_{(ind)(tc)}}{1 - \tan (C)A_{(Vert)(tc)} \tan (DC)_{(ind)(tc)}} \tag{II-7}$$

This is a theoretically correct expression, if no vertical accelerations are present, or if vertical accelerations are assumed to cause a variation in g_{IR}. However, if the size of $(C)A_{(vert)(tc)}$ does not become greater than 10 milliradians, an error of no more than one part in 30,000 is introduced when $(C)A_{(vert)}$ is substituted for $\tan (C)A_{(vert)(tc)}$. This should be true for all cases of practical interest. If this is done, eq (II-7) becomes:

$$(C)\ddot{A}_{(vert)(tc)} + \left(\frac{g_{IR}}{R_E}\right) \frac{(C)A_{(vert)(tc)}}{1-(C)A_{(vert)(tc)} \tan(DC)_{(ind)(tc)}} =$$

$$-\left(\frac{g_{IR}}{R_E}\right) \frac{\tan(DC)_{(ind)(tc)}}{1-(C)A_{(vert)(tc)} \tan(DC)_{(ind)(tc)}} \qquad \text{(II-8)}$$

If $\frac{g_{IR}}{R_E}$ is defined as W^2_{nE}, then eq (II-8) becomes:

$$(C)\ddot{A}_{(vert)(tc)} + W^2_{nE} \frac{(C)A_{(vert)(tc)}}{1-(C)A_{(vert)(tc)} \tan(DC)_{(ind)(tc)}} =$$

$$- W^2_{nE} \frac{\tan(DC)_{(ind)(tc)}}{1-(C)A_{(vert)(tc)} \tan(DC)_{(ind)(tc)}} \qquad \text{(II-9)}$$

If $(DC)_{(ind)(tc)}$ is smaller than fifteen degrees,* $(DC)_{(ind)(tc)}$ can be substituted for $\tan(DC)_{(ind)(tc)}$ for first order theoretical solutions. Also, $(C)A_{(vert)(tc)} \tan(DC)_{(ind)(tc)}$, as the product of two small angles, becomes very much less than unity, and can be neglected. Then:

$$(C)\ddot{A}_{(vert)(tc)} + W^2_{nE} (C)A_{(vert)(tc)} = -W^2_{nE}(DC)_{(ind)(tc)} \qquad \text{(II-10)}$$

Using the same small angle assumptions, and the additional assumption that $A_{Z(cm)}$ remains less than fifteen degrees, eq (II-4) can be written in the approximate form:

$$V_{[E-(cm)](tc)} = V_{[E-(air)](tc)} - V_{[(air)-M]} A_{Z(cm)} \qquad \text{(II-11)}$$

---

* Ref par 3 App A

Equations (II-2), (II-10) and (II-11) are the basic kinematic equations that are used in all the derivations of Chapter III.

3. Longitudinal Motion Kinematic Equations*

The longitudinal problem is one of range indication. The geometrical elements of the simplified range indication problem are illustrated in Fig II-6. Figure II-7 defines the directions and angles that are used in the derivation of the kinematic equations.

As can be seen from Fig II-7, the following relations are true:

$$(DC)_{(true)(long)} = (C)A_{(Vert)(long)} + (DC)_{(ind)(long)} \tag{II-12}$$

$$\ddot{A}_{r(true)} = \frac{a_{[E-(cm)](long)}}{R_E} \tag{II-13}$$

$$\tan(DC)_{(true)(long)} = \frac{a_{[E-(cm)](long)}}{g_{IR}} \tag{II-14}$$

Taking the tangent of eq (II-12)

$$\tan(DC)_{(true)(long)} = \frac{\tan(C)A_{(vert)(long)} + \tan(DC)_{(ind)(long)}}{1-\tan(C)A_{(vert)(long)}\tan(DC)_{(ind)(long)}} \tag{II-15}$$

Equation (II-15) is a theoretically correct expression but, as in the lateral case, all practical conditions find $(C)A_{(vert)(long)}$ smaller than ten milliradians. Then, eq (II-15) simplifies essentially to:

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*This development follows that of John Hutzenlaub, Notebook, July, 1947.

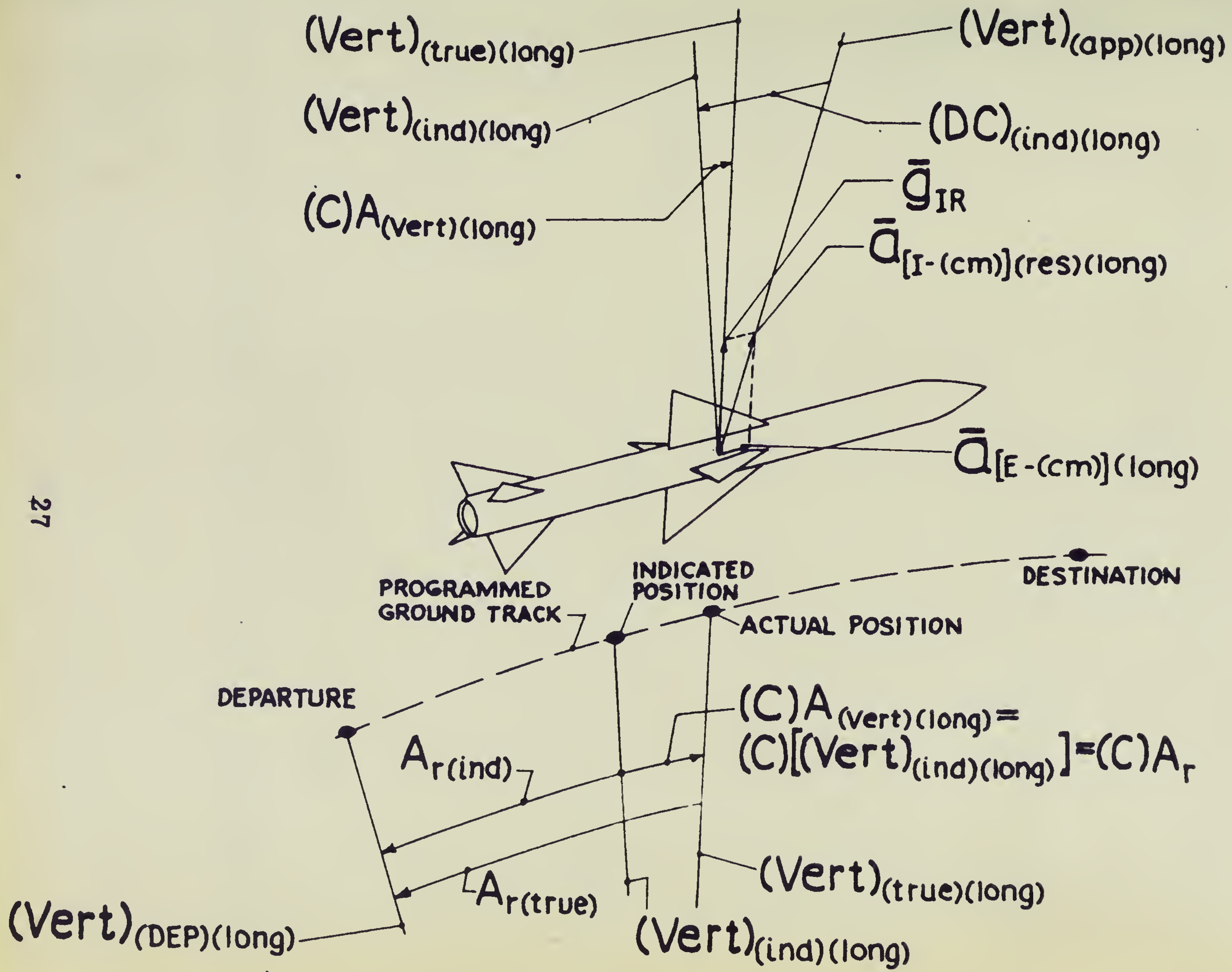

FIG.II-6 GEOMETRICAL ELEMENTS OF RANGE INDICATION PROBLEM.

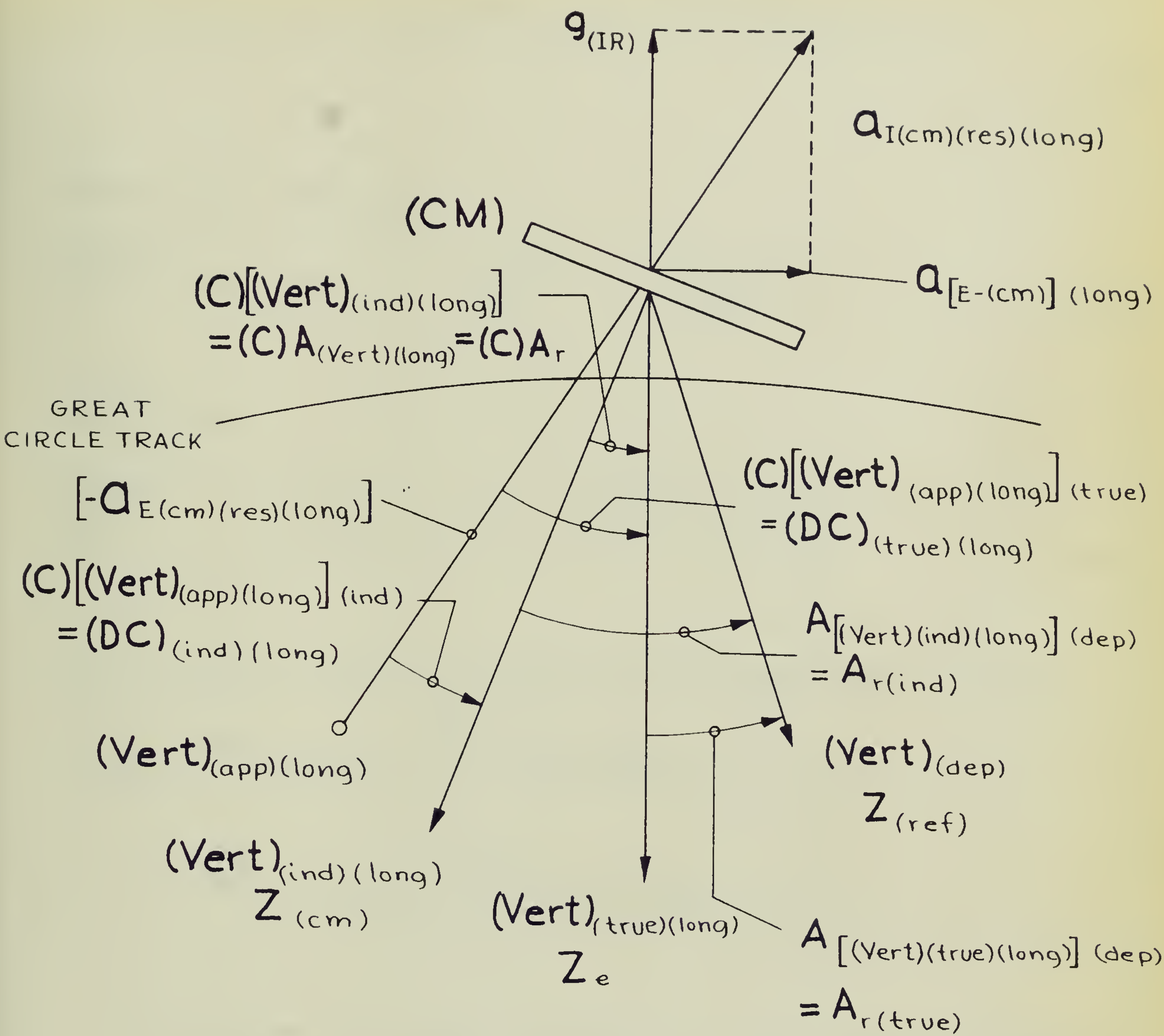

FIG. II-7
LONGITUDINAL INDICATION DIAGRAM

$$\tan(DC)_{(true)(long)} = \frac{(C)A_{(Vert)(long)} + \tan(DC)_{(ind)(long)}}{1-(C)A_{(Vert)(long)}\tan(DC)_{(ind)(long)}} \qquad \text{(II-16)}$$

From eqs (II-13) and (II-14),

$$\tan(DC)_{(true)(long)} = \frac{\ddot{A}_{r(true)}R_E}{g_{IR}} = \frac{\ddot{A}_{r(true)}}{W_{nE}^2} \qquad \text{(II-17)}$$

Substituting eq (II-17) into eq (II-16)

$$\frac{\ddot{A}_{r(true)}}{W_{nE}^2} = \frac{(C)A_{(Vert)(long)} + \tan(DC)_{(ind)(long)}}{1-(C)A_{(Vert)(long)}\tan(DC)_{(ind)(long)}} \qquad \text{(II-18)}$$

From this there results, as the general longitudinal kinematic equation,

$$\ddot{A}_{r(true)} = W_{nE}^2 \frac{(C)A_{(Vert)(long)}}{1-(C)A_{(Vert)(long)}\tan(DC)_{(ind)(long)}} + W_{nE}^2 \frac{\tan(DC)_{(ind)(long)}}{1-(C)A_{(Vert)(long)}\tan(DC)_{(ind)(long)}} \qquad \text{(II-19)}$$

When the same small angle assumptions are made for the longitudinal kinematic relationship that were made in writing eq (II-10) in the lateral case, eq (II-19) reduces to

$$\ddot{A}_{r(true)} = W_{nE}^2 (C)A_{(Vert)(long)} + W_{nE}^2 (DC)_{(ind)(long)} \qquad \text{(II-20)}$$

This equation and the relationships

$$A_{r(true)} = A_{r(ind)} - (C)A_{(vert)(long)} \tag{II-21}$$

and

$$A_{r(app)} = A_{r(ind)} + (DC)_{(ind)(long)} \tag{II-22}$$

form the basic kinematic equations that are used in all the derivations of Chapter IV.

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CHAPTER III

THE TRACK CONTROL PROBLEM

1. Introduction

The simplified kinematic equations developed in Chapter II for the track control problem are restated here for convenience.

$$A_{[E-(cm)](tc)} = R_E (C)\ddot{A}_{(Vert)(tc)} \qquad \text{(III-1)}$$

$$(C)\ddot{A}_{(Vert)(tc)} + W^2_{NE}\ (C)A_{(Vert)(tc)} = -W^2_{NE}\ (DC)_{(ind)(tc)} \qquad \text{(III-2)}$$

$$V_{[E-(cm)](tc)} = V_{[E-(air)](tc)} - V_{[(air)-M]}\ A_{z(cm)} \qquad \text{(III-3)}$$

Using these simplified kinematic equations, various mechanization equations are investigated in an effort to determine a physically realizable and reasonably accurate mechanization of the track control problem. As previously stated, for the purposes of this study, the track control problem is considered as decoupled from the longitudinal problem[1], and vertical accelerations are neglected[2]. It is further assumed that the missile operates with perfect control surface servomechanisms and that

1 - ref par 2 App A

2 - ref par 4 App A

it possesses perfect aerodynamic response[3]. Finally, the pendulous accelerometers are considered to indicate instantaneously the direction of the resultant acceleration[4].

The purpose of the track control mechanization is to reduce toward zero any angle (C) $[(\mathrm{Vert})_{(\mathrm{ind})}]_{(\mathrm{tc})}$ that has been introduced by lateral motion of the missile away from the prescribed great circle track, and to maintain this angle sufficiently small to insure arrival within one mile of the destination. The control is achieved through operations upon signals received from the measurement of the available angle (C) $[(\mathrm{Vert})\mathrm{app}]_{(\mathrm{ind})}$, the pendulum angle, in order to correct the heading angle, $A[X_{(\mathrm{cm})} - (\mathrm{CD})]$.

Throughout the derivations which follow, it is convenient to refer to Fig II-5, which indicates the relationships among the important angles.

2. Proportional Control

The simplest possible mechanization is proportional control, in which the heading of the missile is changed in linear relation to the pendulum angle as measured between the pendulum and the controlled member. This case is therefore considered first. The mechanization equation for this concept can be expressed as:

$$A_{Z(\mathrm{cm})} = -S_{[C(\mathrm{CD})][AA]}\,(\mathrm{DC})_{(\mathrm{ind})(\mathrm{tc})} \tag{III-4}$$

(3) ref par 5 App A

(4) ref par 6 App A

Using this mechanization equation, and the simplified kinematic equations, numbered one through three, a performance equation for this system is now derived.

From eq (III-3),

$$A_{z(cm)} = \frac{V_{[E-(air)](tc)} - V_{[E-(cm)](tc)}}{V_{[(air)-M]}} \qquad \text{(III-5)}$$

Integrating eq (III-1), assuming the constant of integration as zero[1], and substituting into eq (III-5),

$$A_{z(cm)} = \frac{V_{[E-(air)](tc)} - R_E (C)\dot{A}_{(Vert)(tc)}}{V_{[(air)-M]}} \qquad \text{(III-6)}$$

From eq (III-2),

$$-(DC)_{(ind)(tc)} = \frac{(C)\ddot{A}_{(Vert)(tc)}}{W^2_{NE}} + (C)A_{(Vert)(tc)} \qquad \text{(III-7)}$$

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1 - par 7 App A

Substituting (III-6) and (III-7) into the mechanization eq (III-4),

$$\frac{V_{[E-(air)](tc)} - R_E (C)\dot{A}_{(Vert)(tc)}}{V_{[(air)-M]}} = S_{[c(cD)][AA]} \left\{ \frac{(C)\ddot{A}_{(Vert)(tc)}}{W_{NE}^2} + (C)A_{(Vert)(tc)} \right\} \qquad \text{(III-8)}$$

From this, the performance equation can be written:

$$(C)\ddot{A}_{(Vert)(tc)} + \frac{R_E W_{NE}^2}{V_{[(air)-M]} S_{[c(cD)][AA]}} (C)\dot{A}_{(Vert)(tc)} + W_{NE}^2 (C)A_{(Vert)(tc)} = \frac{W_{NE}^2}{S_{[c(cD)][AA]}} \frac{V_{[E-(air)](tc)}}{V_{[(air)-M]}} \qquad \text{(III-9)}$$

Although this equation provides a stable control system, a forced error results from the lateral component of the velocity of the wind, $V_{[E - (air)](tc)}$. This, obviously, is unsatisfactory for the control of guided missiles, because no large steady-state errors can be tolerated.

3. Integral Control

Next, let us examine a system that changes the heading of the missile in accordance with the integral of the pendulum angle. The equation for this mechanization can be written

$$A_{z(cm)} = -S_{[c(cD)][A\dot{A}]} \int (DC)_{(ind)(tc)} \, dt \qquad \text{(III-10)}$$

Differentiating (III-6), and substituting with (III-7) into (III-10) differentiated,

$$\frac{\dot{V}_{[E-(air)](tc)} - R_E (C)\ddot{A}_{(Vert)(tc)}}{V_{[(air)-M]}} = S_{[C(CD)][A\dot{A}]} \left\{ \frac{(C)\ddot{A}_{(Vert)(tc)}}{W^2_{NE}} + (C)A_{(Vert)(tc)} \right\} \qquad (III-11)$$

or

$$(C)\ddot{A}_{(Vert)(tc)} + \frac{W^2_{NE}}{\dfrac{g_{IR}}{S_{[C(CD)][A\dot{A}]} V_{[(air)-M]}} + 1} (C)A_{(Vert)(tc)} = \frac{W^2_{NE}}{g_{IR} + S_{[C(CD)][A\dot{A}]} V_{[(air)-M]}} V_{[E-(air)](tc)} \qquad (III-12)$$

This system, although the forced error is now due to the acceleration of the wind, has no damping term. $S_{[(C)(CD)][A\dot{A}]}$ can be made large to reduce the size of the forced error, but this results in an undamped period of oscillation of approximately 84 minutes for the controlled member. Any residual forced error (since it is physically impossible to make $S_{[(C)(CD)][A\dot{A}]}$ infinite), causes this oscillation to appear.

4. Second Integral Control

In order to reduce the forced error, consider next a system that changes the missile heading in obedience to the second integral of the pendulum angle. Here, the mechanization equation is:

$$A_{Z(cm)} = -S_{[C(CD)][A\ddot{A}]} \iint (DC)_{(ind)(tc)}\, dt\, dt \qquad (III-13)$$

Differentiating (III-6) twice, and substituting with (III-7) into (III-13), differentiated twice,

$$\frac{V_{[E-(air)](tc)} - R_E\,(C)\overset{\dots}{\ddot{A}}_{(Vert)(tc)}}{V_{[(air)-M]}} = \dot{S}_{[c(CD)]\,[A\ddot{A}]} \left\{ \frac{(C)\ddot{A}_{(Vert)(tc)}}{W_{NE}^{2}} + (C)A_{(Vert)(tc)} \right\} \tag{III-14}$$

or,

$$(C)\overset{\dots}{\ddot{A}}_{(Vert)(tc)} + S_{[c(CD)][A\ddot{A}]}\,\frac{V_{[(air)-M]}}{g_{IR}}\,(C)\ddot{A}_{(Vert)(tc)} + S_{[c(CD)]\cdot[A\ddot{A}]}\,\frac{V_{[(air)-M]}}{R_E}\,(C)A_{(Vert)(tc)} = \frac{1}{R_E}\,\ddot{V}_{[E-(air)](tc)} \tag{III-15}$$

By Routh's stability criteria[1], this is unconditionally unstable.

5. Derivative Control

In order to give a complete presentation of possible types of mechanization, consider the heading angle to be controlled in accordance with the derivative of the pendulum angle:

$$\int A_{z(cm)}\,dt = -S_{[c(CD)]\,[\dot{A}A]}\,(DC)_{(ind)(tc)} \tag{III-16}$$

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1 - Routh, "Advanced Rigid Dynamics".

Differentiating (III-7), and substituting, with (III-6) into (III-16), differentiated,

$$\frac{V_{[E-(air)](tc)} - R_E (C)\dot{A}_{(Vert)(tc)}}{V_{[(air)-M]}} = S_{[c(CD)][\dot{A}A]} \left\{ \frac{(C)\ddot{A}_{(Vert)(tc)}}{W_{NE}^2} + (C)\dot{A}_{(Vert)(tc)} \right\} \qquad \text{(III-17)}$$

From (III-17)

$$(C)\dddot{A}_{(Vert)(tc)} + W_{NE}^2 \left\{ 1 + \frac{R_E}{S_{[c(CD)][\dot{A}A]} V_{[(air)-M]}} \right\} (C)\dot{A}_{(Vert)(tc)} = \frac{V_{[E-(air)](tc)}}{V_{[(air)-M]}} \quad \frac{W_{NE}^2}{S_{[c(CD)][\dot{A}A]}} \qquad \text{(III-18)}$$

By Routh's stability criteria for a cubic, this is unconditionally unstable. Moreover, there is a forced error due to the velocity of the wind.

6. Proportional Plus Integral Control

As shown in paragraph 2, proportional control produces a solution which, although it has a forced error caused by wind velocity, is nevertheless stable, while integral control (paragraph 3) develops a solution that does not have any forced error caused by wind, but which also has no damping, and is therefore on the verge of instability. Thus it might be expected that by combining both equations, a stable system without forced error due to velocity of wind might result. The resulting mechanization equation is:

$$A_{z(cm)} = -S_{[c(CD)][\dot{A}\dot{A}]} (DC)_{(ind)(tc)} - S_{[c(CD)][A\dot{A}]} \int (DC)_{(ind)(tc)}\, dt \qquad \text{(III-19)}$$

Differentiating eqs (III-6) and (III-7) and substituting with eqs (III-7) into (III-19) differentiated,

$$\frac{\dot{V}_{[E-(air)](tc)} - R_E (C)\ddot{A}_{(Vert)(tc)}}{V_{[(air)-M]}} = S_{[C(CD)][\dot{A}\dot{A}]} \left\{ \frac{(C)\ddot{A}_{(Vert)(tc)}}{W^2_{NE}} + (C)\dot{A}_{(Vert)(tc)} \right\} + S_{[C(CD)][A\dot{A}]} \left\{ \frac{(C)\ddot{A}_{(Vert)(tc)}}{W^2_{NE}} + (C)A_{(Vert)(tc)} \right\} \tag{III-20}$$

From eq (III-20),

$$(C)\ddot{A}_{(Vert)(tc)} + \frac{W^2_{NE}}{S_{[C(CD)][\dot{A}\dot{A}]}} \left\{ \frac{R_E}{V_{[(air)-M]}} + \frac{S_{[C(CD)][A\dot{A}]}}{W^2_{NE}} \right\} (C)\ddot{A}_{(Vert)(tc)} + W^2_{NE}\,(C)\dot{A}_{(Vert)(tc)} + W^2_{NE}\,\frac{S_{[C(CD)][A\dot{A}]}}{S_{[C(CD)][\dot{A}\dot{A}]}}(C)A_{(Vert)(tc)} = \frac{W^2_{NE}}{S_{[C(CD)][\dot{A}\dot{A}]}\,V_{[(air)-M]}}\,\dot{V}_{[E-(air)]} \tag{III-21}$$

By Routh's criteria of stability, this system is stable if:

(a) $S_{[(C)(CD)][\dot{A}\dot{A}]}$ and $S_{[(C)(CD)][A\dot{A}]}$ are positive numbers,

(b) $\dfrac{W^2_{NE}}{S_{[(C)(CD)][\dot{A}\dot{A}]}\,V_{[(air)-M]}} > 0.$

Furthermore, as expected, this system has no forced error resulting from wind velocity, although it has one caused by the acceleration of the wind. This suggests adding second integral to the system just discussed.

7. Proportional Plus First and Second Integral Control

$$A_{z(cm)} = -S_{[c(cD)][\dot{A}\ddot{A}]}(DC)_{(ind)(tc)} - S_{[c(cD)][\dot{A}\ddot{A}]}\int (DC)_{(ind)(tc)}\,dt - S_{[c(cD)][\dot{A}\ddot{A}]}\iint (DC)_{(ind)(tc)}\,dt\,dt \qquad \text{(III-22)}$$

Substituting eqs (III-6) and (III-7) differentiated twice, eq (III-7) differentiated, and eq (III-7) into eq (III-22) differentiated twice,

$$\frac{\ddot{V}_{[E-(air)](tc)} - R_E(C)\ddot{A}_{(vert)(tc)}}{V_{[(air)-M]}} = S_{[c(cD)][\dot{A}\ddot{A}]}\left\{\frac{(C)\dddot{A}_{(vert)(tc)}}{W_{NE}^2} + (C)\ddot{A}_{(vert)(tc)}\right\} + S_{[c(cD)][\dot{A}\ddot{A}]}\left\{\frac{(C)\ddot{A}_{(vert)(tc)}}{W_{NE}^2} + (C)\dot{A}_{(vert)(tc)}\right\} + S_{[c(cD)][\dot{A}\ddot{A}]}\left\{\frac{(C)\ddot{A}_{(vert)(tc)}}{W_{NE}^2} + (C)A_{(vert)(tc)}\right\} \qquad \text{(III-23)}$$

From eq (III-23)

$$(C)\dddot{A}_{(vert)(tc)} + \left\{\frac{g_{IR}}{S_{[c(cD)][\dot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\dot{A}\ddot{A}]}}\right\}(C)\ddot{A}_{(vert)(tc)} + \left\{W_{NE}^2 + \frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\dot{A}\ddot{A}]}}\right\}(C)\ddot{A}_{(vert)(tc)} + \frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\dot{A}\ddot{A}]}}W_{NE}^2(C)\dot{A}_{(vert)(tc)} + \frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\dot{A}\ddot{A}]}}W_{NE}^2(C)A_{(vert)(tc)} = -\frac{W_{NE}^2}{S_{[c(cD)][\dot{A}\ddot{A}]}V_{[(air)-M]}}\ddot{V}_{[E-(air)]} \qquad \text{(III-24)}$$

According to Routh's stability criteria for a quartic, this system is stable if:

(a) The three sensitivities have the same sign. (They may be all positive or all negative),

(b) $$W_{NE}^2 S_{[c(cD)][\dot{A}\ddot{A}]} - \frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\dot{A}\ddot{A}]}}\left\{\frac{g_{IR}}{V_{[(air)-M]}} + S_{[c(cD)][\dot{A}\ddot{A}]}\right\} > 0$$

In this system no effect of the wind of lower order than the second derivative of the wind velocity enters into the forced error. It seems unlikely that a forced error of this nature could have a magnitude large enough to be of practical importance. However, if this error should prove important, it is possible to go to higher order performance equations. One further case is considered.

8. Proportional Plus First, Second and Third Integral Control

$$A_{z(cm)} = -S_{[c(cD)][\ddot{A}\,\ddot{A}]}(DC)_{(ind)(tc)} - S_{[c(cD)][\ddot{A}\,\ddot{A}]}\int (DC)_{(ind)(tc)}\,dt - S_{[c(cD)][\dot{A}\,\ddot{A}]}\iint (DC)_{(ind)(tc)}\,dt\,dt - S_{[c(cD)][A\,\ddot{A}]}\iiint (DC)_{(ind)(tc)}\,dt\,dt\,dt \qquad (III\text{-}25)$$

Substituting eqs (III-6) and (III-7) differentiated three times, eq (III-7) differentiated twice, eq (III-7) differentiated, and eq (III-7) into eq (III-25) differentiated three times,

$$\frac{\dddot{V}_{[e-(air)][tc]} - R_E (C)\dddot{A}_{(Vert)(tc)}}{V_{[(air)-M]}} = S_{[c(cD)][\ddot{A}\,\ddot{A}]}\left\{\frac{(C)\overset{\cdot\cdot\cdot\cdot\cdot}{A}_{(Vert)(tc)}}{W_{NE}^2} + (C)\dddot{A}_{(Vert)(tc)}\right\} + S_{[c(cD)][\ddot{A}\,\ddot{A}]}\left\{\frac{(C)\ddddot{A}_{(Vert)(tc)}}{W_{NE}^2} + (C)\ddot{A}_{(Vert)(tc)}\right\} + S_{[c(cD)][\dot{A}\,\ddot{A}]}\left\{\frac{(C)\dddot{A}_{(Vert)(tc)}}{W_{NE}^2} + (C)\dot{A}_{(Vert)(tc)}\right\} + S_{[c(cD)][A\,\ddot{A}]}\left\{\frac{(C)\ddot{A}_{(Vert)(tc)}}{W_{NE}^2} + (C)A_{(Vert)(tc)}\right\} \qquad (III\text{-}26)$$

From eq (III-26)

$$(C)\overset{\cdot\cdot\cdot\cdot\cdot}{A}_{(Vert)(tc)} + \left\{\frac{g_{IR}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}\right\}(C)\ddddot{A}_{(Vert)(tc)} + \left\{W_{NE}^2 + \frac{S_{[c(cD)][\dot{A}\,\ddot{A}]}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}\right\}(C)\dddot{A}_{(Vert)(tc)} + \left\{W_{NE}^2\frac{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}} + \frac{S_{[c(cD)][A\,\ddot{A}]}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}\right\}(C)\ddot{A}_{(Vert)(tc)} + \left\{W_{NE}^2\frac{S_{[c(cD)][\dot{A}\,\ddot{A}]}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}\right\}(C)\dot{A}_{(Vert)(tc)} + \left\{W_{NE}^2\frac{S_{[c(cD)][A\,\ddot{A}]}}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}}\right\}(C)A_{(Vert)(tc)} = \frac{W_{NE}^2}{S_{[c(cD)][\ddot{A}\,\ddot{A}]}V_{[(air)-M]}}\dddot{V}_{[E-(air)]} \qquad (III\text{-}27)$$

By Routh's stability criteria for a quintic, this performance equation is stable if:

(a) All coefficients of the homogeneous equation are positive,

(b)

$$\frac{S_{[c(CD)][\dot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}W_{NE}^{2} - \left[\frac{\left\{\frac{S_{[c(CD)][\dot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right\}W_{NE}^{2}}{\left\{\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right\}}\right] -$$

$$\left[\frac{\left\{\left(\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right)\left(W_{NE}^{2} + \frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right) - \left(W_{NE}^{2}\frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}} + \frac{S_{[c(CD)][\dot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right)\right\}^{2}\left\{W_{NE}^{2}\frac{S_{[c(CD)][\dot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right\}}{\left\{\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right\}\left\{W_{NE}^{2}\frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}} + \frac{S_{[c(CD)][\dot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right\}\left\{\left(\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right)\left(W_{NE}^{2} + \frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right) - \left(W_{NE}^{2}\frac{S_{[c(CD)][\ddot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}} + \frac{S_{[c(CD)][\dot{A}\dddot{A}]}}{S_{[c(CD)][\ddot{A}\dddot{A}]}}\right)\right\} - \left\{\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)}}{S_{[c(CD)}}\right.}\right.$$

These restrictions permit the sensitivities to be either positive or negative, provided that they all have the same sign.

It is of interest to notice that in each stable performance equation, the forced error has the same type of coefficient, with the order of the wind derivative increasing from velocity for proportional control to any derivative desired, as the performance equation increases in complexity. This coefficient is of the form:

$$
\frac{\frac{S_{[c(CD)][\ddot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}} + \frac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\Big)\Big\}^{2}\left\{W_{NE}^{2}\frac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right\}}{\left\{\left(\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right)\left(W_{NE}^{2} + \frac{S_{[c(CD)][\dot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right) - \left(W_{NE}^{2}\frac{S_{[c(CD)][\ddot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}} + \frac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right)\right\} - \left\{\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right\}\left\{\left(\frac{g_{IR}}{S_{[c(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right)\left(\frac{S_{[c(CD)][\dot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}W_{NE}^{2}\right) - \left(\frac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}W_{NE}^{2}\right)\right\}}\Bigg] > 0
$$

$$\frac{W_{NE}^{2}}{V_{[(air)-M]}\, S_{[c(CD)][AA]}}$$

The forced error is thus seen to decrease as the missile airspeed increases, and as the sensitivity $S_{[C(CD)][AA]}$ increases. Theoretically, if this sensitivity increased to infinity, there would be no error, either transient or in the steady state, caused by the action of the wind. This is similar to the conditions found with integral control, eq (III-12), where, as the sensitivity $S_{[C(CD)][AA]}$ increases to infinity, both steady state and transient wind errors reduce to zero.

9. Track Control Correction as a Function of Wind

Since the mechanization eq (III-22) for proportional plus first and second integral control appears to provide a stable system without excessive forced error, this system is considered further. The performance equation for this system, eq (III-24), can be expressed in operational form:

$$\frac{(C)A_{(Vert)(tc)}}{\dot{V}_{[E-(air)]}} = \frac{\left\{\dfrac{W_{NE}^{2}}{S_{[c(CD)][\ddot{A}\ddot{A}]}\; V_{[(air)-M]}}\right\} p}{p^{4} + \left\{\dfrac{g_{IR}}{S_{[c(CD)][\ddot{A}\ddot{A}]}\; V_{[(air)-M]}} + \dfrac{S_{[c(CD)][\dot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right\} p^{3} + \left\{W_{NE}^{2} + \dfrac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right\} p^{2} + \left\{\dfrac{S_{[c(CD)][\dot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}} W_{NE}^{2}\right\} p + \left\{\dfrac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}} W_{NE}^{2}\right\}} \qquad \text{(III-28)}$$

If $(C)A_{(vert)}$ is expressed in minutes it is a measure, in nautical miles, of the transverse linear distance of the missile from the desired great circle track. The presence of the operational symbol in the numerator of the equation indicates that there is no error, in the steady state, from an accelerating wind. If it is desired to find the missile response, $A_{Z(cm)}$, to an accelerating wind, it can be calculated if eq (III-28) is multiplied by an expression for $\frac{A_{Z(cm)}}{(C)A_{(vert)(tc)}}$. The result yields the heading of the missile as a function of wind velocity. Manipulating eqs (III-1), (III-2), (III-3), and (III-22), and expressing the result in operational form,

$$\frac{A_{z(cm)}}{(C)A_{(vert)(tc)}} = -\frac{\left\{\frac{S_{[c(cD)][\ddot{A}\ddot{A}]}R_E}{g_{IR}}\right\}p^4 + \left\{\frac{S_{[c(cD)][\ddot{A}\ddot{A}]}R_E}{g_{IR}}\right\}p^3 + \left\{\frac{S_{[c(cD)][A\ddot{A}]}R_E}{g_{IR}} + S_{[c(cD)][\ddot{A}\ddot{A}]}\right\}p^2 + \left\{S_{[c(cD)][\ddot{A}\ddot{A}]}\right\}p + S_{[c(cD)][A\ddot{A}]}}{p^2} \qquad (III\text{-}29)$$

Combining eqs (III-28) and (III-29) gives,

$$\frac{A_{z(cm)}}{\dot{V}_{[E-(air)]}} = -\frac{\left\{\frac{1}{V_{[(air)-M]}}\right\}p^4 + \left\{\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right\}p^3 + \left\{\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{W_{NE}^2}{V_{[(air)-M]}}\right\}p^2 + \left\{\frac{S_{[c(cD)][\ddot{A}\ddot{A}]}W_{NE}^2}{S_{[c(cD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right\}p + \left\{\frac{S_{[c(cD)][A\ddot{A}]}W_{NE}^2}{S_{[c(cD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right\}}{p\left[p^4 + \left\{\frac{g_{IR}}{S_{[c(cD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(cD)][\ddot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right\}p^3 + \left\{W_{NE}^2 + \frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right\}p^2 + \left\{\frac{S_{[c(cD)][\ddot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}W_{NE}^2\right\}p + \left\{\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}W_{NE}^2\right\}\right]} \qquad (III$$

The negative sign shows that the heading $A_{Z(cm)}$ of the airplane is correct to oppose the tendency of $\dot{V}_{[E - (air)]}$ to blow the missile off the great circle course. This is true even if the sensitivities are negative.

10. Track Control Closed Loop System

The relationship of eq (III-30) can be obtained from a study of the over-all lateral system as a servomechanism. In the direct servo closed-loop system, the angle $(C)A_{(vert)}$ does not appear, but it can easily be obtained, as indicated in the accompanying Fig III-1.

In this figure:

$(PF)_1$ has the value $\dfrac{1}{g_{IR}}$

$(PF)_2 = \dfrac{A_{Z(cm)}}{(DC)_{(ind)(tc)}}$ and is made up of the sum of two performance functions,

$(PF)_a = \dfrac{(DC)_{(ind)(tc)}}{(DC)_{(true)(tc)}}$ and

$(PF)_b = \dfrac{A_{Z(cm)}}{(DC)_{(ind)(tc)}}$ which is the mechanization equation in operational form

$(PF)_3$, in the feedback path, is $\left(-\dfrac{V_{[(air) - M]}}{g_{IR}}\right)p$, providing the proper function for the combinor, which is physically the pendulous unit.

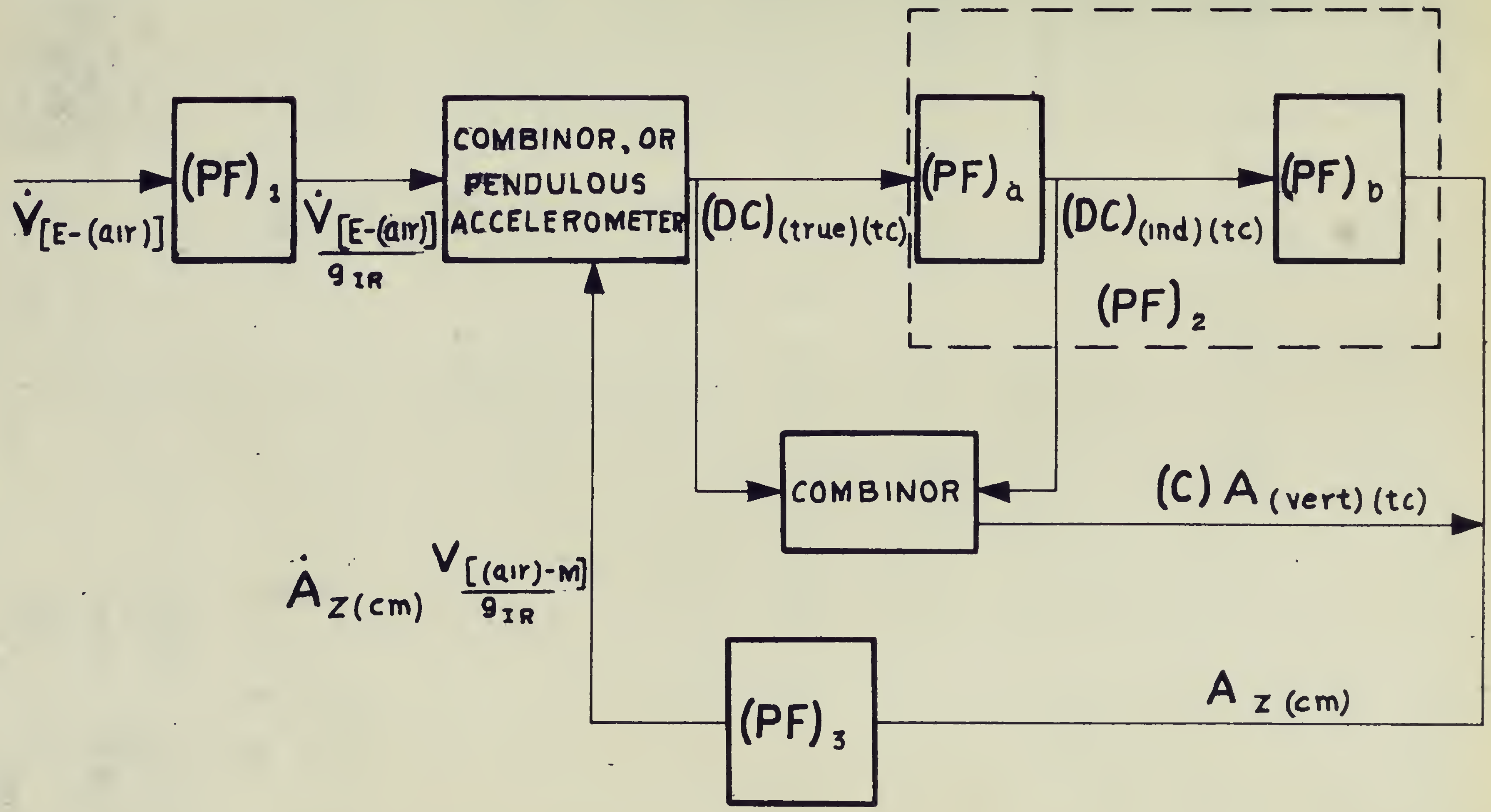

FIG. III - 1. TRACK CONTROL SYSTEM AS A CLOSED LOOP SYSTEM.

The pendulous unit, if $(DC)_{(true)}$ remains smaller than about fifteen degrees, solves the equation

$$\frac{V_{[(air)-M]}}{g_{IR}}\dot{A}_{z(cm)} - \frac{\dot{V}_{[E-(air)]}}{g_{IR}} = (DC)_{(true)(tc)} \qquad (III\text{-}31)$$

This equation is derived from eqs (III-1), (III-2), (III-3) differentiated, and the relationship that

$$(DC)_{(true)(tc)} = (C)A_{(Vert)(tc)} + (DC)_{(ind)(tc)} \qquad (III\text{-}32)$$

This last equation can readily be obtained from an inspection of Fig II-1.

Solving for the performance function $(PF)_a$, using eqs (III-1), (III-2), (III-3), and (III-32),

$$(PF)_a = \frac{(DC)_{(ind)(tc)}}{(DC)_{(true)(tc)}} = \frac{p^2 + W_{NE}^2}{p^2} \qquad (III\text{-}33)$$

The mechanization equation, which in this example is eq (III-22) expressed in operational form, is

$$(PF)_b = \frac{A_{z(cm)}}{(DC)_{(ind)(tc)}} = -\frac{S_{[c(cD)][\ddot{A}\ddot{A}]}\left\{p^2 + \frac{S_{[c(cD)][\dot{A}\dot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}p + \frac{S_{[c(cD)][A\dot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right\}}{p^2} \qquad (III\text{-}34)$$

The product of eqs (III-33) and (III-34) gives

$$(PF)_2 = \frac{A_{z(cm)}}{(DC)_{(true)(tc)}} = -\frac{S_{[c(CD)][\ddot{A}\ddot{A}]}\left\{p^2 + \frac{S_{[c(CD)][\dot{A}\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\,p + \frac{S_{[c(CD)][A\ddot{A}]}}{S_{[c(CD)][\ddot{A}\ddot{A}]}}\right\}\left\{\frac{g_{IR}}{R_E} + p^2\right\}}{p^4} \qquad \text{(III-35)}$$

but, from the theory of servomechanisms,[1]

$$\frac{A_{z(cm)}}{\dot{V}_{[E-(air)]}} = \frac{(PF)_1 (PF)_2}{1 + (PF)_2 (PF)_3} \qquad \text{(III-36)}$$

Solving this equation gives eq (III-30), as before. The stability criteria for eq (III-30) are identical with those for eq (III-24). Since the sensitivities can have either sign, a necessary condition for stability is that $\left| S_{[C(CD)][\dot{A}\ddot{A}]} \right| > \left| \frac{g_{IR}}{V_{[E-(air)]}} \right|$,

if the sensitivities are chosen as negative. If this is expressed as

$$S_{[C(CD)][\dot{A}\ddot{A}]} = -K\frac{g_{IR}}{V_{[E-(air)]}} \quad (\text{where } K > 1 \text{ and } S < 0), \qquad \text{(III-37)}$$

---

1 - Brown and Campbell, "Principles of Servomechanisms".

then, for stability,

$$K > \frac{1}{1 - \frac{S_{[c(c\,D)][\ddot{A}\,\ddot{A}]}}{S_{[c(c\,D)][A\,\ddot{A}]}} W_{NE}^{2}} \tag{III-38}$$

11. Numerical Solutions, Using Two Quadratics

Using the relationships that have been established, and substituting numerical values for the coefficients, it is possible to solve for the various angles as a function of the acceleration of the wind $\dot{V}_{[E\,-\,(air)]}$. From a sequence of such solutions, using a range of values for the sensitivities, it is possible to select the most satisfactory values, and to estimate the effectiveness of the system. This is now done.

The homogeneous portion of the performance equation,(III-24), is a biquadratic. It can then be written

$$(p^2 + 2(DR)_1 W_1 + W_1^2)(p^2 + 2(DR)_2 W_2 + W_2^2) = 0 \tag{III-39}$$

which factors the equation into two quadratics. Manipulation of the equation indicates that if the damping ratios and natural frequencies are varied, the higher damping term associates itself with the shorter of the two frequencies. This result then appears essentially as that of a lightly damped long period quadratic, which is undesirable. For this reason $(DR)_1$ and $(DR)_2$, and W_1 and W_2 have been chosen equal for all solutions of the equation,

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and equations are now derived to facilitate the selection of values for the sensitivities to make these limitations hold.

Expanding (III-39), in the general form, gives

$$p^4+\left[2(DR)_1W_1+2(DR)_2W_2\right]p^3+\left[W_1^2+W_2^2+4(DR)_1(DR)_2W_1W_2\right]p^2+$$
$$\left[2(DR)_1W_1^2W_2+2(DR)_2W_1W_2^2\right]p+W_1^2W_2^2=0 \qquad \text{(III-40)}$$

If $\dfrac{W_1}{W_{NE}}=\dfrac{W_2}{W_{NE}}=(FR)$, and $(DR)_1=(DR)_2=(DR)$ then,

comparing the coefficients of eq (III-40) with those of eq (III-24) gives the following relationships:

$$\left[\frac{g_{IR}}{S_{[c(cD)][\ddot{A}\ddot{A}]}\,V_{[(air)-M]}}+\frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right]=4\,(DR)(FR)W_{NE}\,, \qquad \text{(III-41)}$$

$$\left[W_{NE}^2+\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right]=2\,(FR)^2\,W_{NE}^2\left[2(DR)^2+1\right], \qquad \text{(III-42)}$$

$$\frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}W_{NE}^2=4(FR)^3(DR)\,W_{NE}^3\,, \qquad \text{(III-43)}$$

$$\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}W_{NE}^2=(FR)^4\,W_{NE}^4 \qquad \text{(III-44)}$$

When these four equations are solved simultaneously for the sensitivities and for the damping ratio, the results are:

$$(DR) = \frac{(FR)^2 - 1}{2(FR)} \qquad (III\text{-}45)$$

$$S_{[c(CD)][\ddot{A}\ddot{A}]} = \frac{g_{IR}}{8\, W_{NE}\, V_{[(air)-M]}\, (DR)^2 (FR)^2} \qquad (III\text{-}46)$$

$$S_{[c(CD)][\dot{A}\ddot{A}]} = \frac{g_{IR}\,(FR)}{2V_{[(air)-M]}\,(DR)} \qquad (III\text{-}47)$$

$$S_{[c(CD)][A\ddot{A}]} = \frac{g_{IR}\, W_{NE}\, (FR)^2}{8V_{[(air)-M]}\,(DR)^2} \qquad (III\text{-}48)$$

It is noticed that the choice of a damping ratio immediately fixes all sensitivities and the frequency ratio (or that the choice of a frequency ratio, alternatively, fixes the damping ratio). As the frequency ratio increases; i.e., as the quadratics have progressively higher natural frequencies, the damping ratio simultaneously increases without limit. Conversely, as the frequency of the quadratics approaches that of the 84 minute period "earth pendulum", W_{NE} , the damping reduces to zero. The sensitivities, at the same time, increase without limit.

12. Numerical Solution Using Four First Order Terms

A second method of writing the quartic equation, which can be examined easily, and which may yield results of practical interest, is that in which the equation is reduced to four decaying exponentials. This equation is of the form

$$\left(p+\frac{1}{(CT)_1}\right)\left(p+\frac{1}{(CT)_2}\right)\left(p+\frac{1}{(CT)_3}\right)\left(p+\frac{1}{(CT)_4}\right)=0 \qquad \text{(III-49)}$$

When expanded, this becomes

$$p^4+\left[\frac{1}{(CT)_1}+\frac{1}{(CT)_2}+\frac{1}{(CT)_3}+\frac{1}{(CT)_4}\right]p^3+\left[\frac{1}{(CT)_1(CT)_2}+\frac{1}{(CT)_1(CT)_3}+\frac{1}{(CT)_1(CT)_4}+\frac{1}{(CT)_2(CT)_3}+\frac{1}{(CT)_2(CT)_4}\right.$$

$$\left.+\frac{1}{(CT)_3(CT)_4}\right]p^2+\left[\frac{1}{(CT)_1(CT)_2(CT)_3}+\frac{1}{(CT)_1(CT)_2(CT)_4}+\frac{1}{(CT)_1(CT)_3(CT)_4}+\frac{1}{(CT)_2(CT)_3(CT)_4}\right]p$$

$$+\frac{1}{(CT)_1(CT)_2(CT)_3(CT)_4}=0 \qquad \text{(III-50)}$$

The general relationships among the coefficients of eq (III-50) and eq (III-24) give

$$\left[\frac{g_{IR}}{S_{[c(CO)][\ddot{A}\ddot{A}]}\,V_{[(air)-M]}}+\frac{S_{[c(CO)][\dot{A}\ddot{A}]}}{S_{[c(CO)][\ddot{A}\ddot{A}]}}\right]=\frac{1}{(CT)_1+(CT)_2+(CT)_3+(CT)_4} \qquad \text{(III-51)}$$

$$\left[W_{NE}^2+\frac{S_{[c(CO)][\dot{A}\ddot{A}]}}{S_{[c(CO)][\ddot{A}\ddot{A}]}}\right]=\frac{1}{(CT)_1(CT)_2+(CT)_1(CT)_3+(CT)_1(CT)_4+(CT)_2(CT)_3+(CT)_2(CT)_4+(CT)_3(CT)_4} \qquad \text{(III-52)}$$

$$\frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}} W_{NE}^{2} = \frac{1}{(CT)_1(CT)_2(CT)_3 + (CT)_1(CT)_2(CT)_4 + (CT)_1(CT)_3(CT)_4 + (CT)_2(CT)_3(CT)_4} \qquad \text{(III-53)}$$

$$\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}} W_{NE}^{2} = \frac{1}{(CT)_1(CT)_2(CT)_3(CT)_4} \qquad \text{(III-54)}$$

The special case which is considered here is that for which $(CT)_1 = (CT)_2 = (CT)_3 = (CT)_4 = (CT)$. Under these special circumstances, the simultaneous solution of eqs (III-51, 52, 53, and 54) yields

$$\frac{1}{(CT)^2} = \frac{W_{NE}^2}{2}\left(6 \pm \sqrt{32}\right), \qquad \text{(III-55)}$$

from which

$$\frac{1}{(CT)} = W_{NE} \;\; 2.418 \;\; \text{or} \;\; \frac{1}{(CT)} = W_{NE} \;\; 0.412. \qquad \text{(III-56)}$$

$$S_{[c(cD)][\ddot{A}\ddot{A}]} = \frac{g_{IR}}{V_{[(air)-M]}} = \left[\frac{W_{NE}^2}{\dfrac{4W_{NE}^2}{(CT)} - \dfrac{4}{(CT)^3}}\right] \qquad \text{(III-57)}$$

$$S_{[c(cD)][\dot{A}\ddot{A}]} = \frac{g_{IR}}{V_{[(air)-M]}} \left[\frac{\dfrac{1}{(CT)^2}}{W_{NE}^2 - \dfrac{1}{(CT)^2}}\right] \qquad \text{(III-58)}$$

$$S_{[c(cD)][A\ddot{A}]} = \frac{g_{1R}}{V_{[(air)-M]}} \left[\frac{\frac{1}{(CT)^3}}{4W_{NE}^2 - \frac{4}{(CT)^2}} \right] \tag{III-59}$$

Evaluating (III-57), (III-58), and (III-59), using eq (III-56) with the smaller (CT),

$$S_{[c(cD)][\ddot{A}\dot{A}]} = -\frac{g_{1R}}{V_{[(air)-M]} W_{NE}\, 45} \tag{III-60}$$

$$S_{[c(cD)][\dot{A}\ddot{A}]} = -\frac{g_{1R}\, 1.205}{V_{[(air)-M]}} \tag{III-61}$$

$$S_{[c(cD)][A\ddot{A}]} = -\frac{g_{1R} W_{NE}}{V_{[(air)-M]}\, 0.728} \tag{III-62}$$

This gives negative values to the sensitivities. The alternative evaluation would make the sensitivities positive, but would result in characteristic times fār longer than could be tolerated. The system, with negative sensitivities, is stable by Routh's criteria.

13. Numerical Solution Using Quadratic and Two First Order Terms

The final method of examining a quartic equation which may be

of interest is that in which the quartic is broken up into one quadratic term, and two decaying exponentials,

$$\left(p^2+2(DR)Wp+W^2\right)\left(p+\frac{1}{(CT)_1}\right)\left(p+\frac{1}{(CT)_2}\right) \tag{III-63}$$

Upon expanding eq (III-63) this becomes,

$$p^4+\left[2(DR)W+\frac{1}{(CT)_1}+\frac{1}{(CT)_2}\right]p^3+\left[W^2+\frac{2(DR)W}{(CT)_1}+\frac{2(DR)W}{(CT)_2}\right]p^2+$$
$$\left[\frac{2(DR)W}{(CT)_1(CT)_2}+\frac{W^2}{(CT)_1}+\frac{W^2}{(CT)_2}\right]p \tag{III-64}$$

The general relationships among the coefficients of equations (III-64) and (III-24) are

$$\left[\frac{g_{IR}}{S_{[c(cD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}+\frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right]=2(DR)W+\frac{1}{(CT)_1}+\frac{1}{(CT)_2}, \tag{III-65}$$

$$\left[W_{NE}^2+\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}\right]=W^2+\frac{2(DR)W}{(CT)_1}+\frac{2(DR)W}{(CT)_2}+\frac{1}{(CT)_1(CT)_2}, \tag{III-66}$$

$$\frac{S_{[c(cD)][\dot{A}\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}W_{NE}^2=\frac{2(DR)W}{(CT)_1(CT)_2}+\frac{W^2}{(CT)_1}+\frac{W^2}{(CT)_2}, \tag{III-67}$$

$$\frac{S_{[c(cD)][A\ddot{A}]}}{S_{[c(cD)][\ddot{A}\ddot{A}]}}W_{NE}^2=\frac{W^2}{(CT)_1(CT)_2} \tag{III-68}$$

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These equations can be solved simultaneously for the sensitivities, and for the relation which connects $(CT)_1$, $(CT)_2$ W and (DR). It is convenient to use the relationships: $(FR) = \frac{W_{NE}}{W}\left[(CT)PR\right] = \frac{(CT)_1 W_{NE}}{2\pi}$ and $\left[(CT)PR\right]_2 = \frac{(CT)_2 W_{NE}}{2\pi}$. The resulting equations are:

$$4\pi^2 (FR)^2 [(CT)PR]_1 [(CT)PR]_2 + 1 = 4\pi^2 [(CT)PR]_1 [(CT)PR]_2 + 4\pi (DR)(FR) \left\{ [(CT)PR]_1 + [(CT)PR]_2 \right\} + 1 \qquad \text{(III-69)}$$

$$S_{[c(CD)][\ddot{A}\ddot{A}]} = \frac{2\pi^2 g_{1R} (FR)^2 [(CT)PR]_1 [(CT)PR]_2}{V_{[(air)-M]} W_{NE} \left\{ 4\pi^2 (DR)(FR)[(CT)PR]_1 [(CT)PR]_2 + \pi (FR)^2 \left([(CT)PR]_1 + [(CT)PR]_2 \right) - (DR)(FR) - \pi \left([(CT)PR]_1 + [(CT)PR]_2 \right) \right\}} \qquad \text{(III-70)}$$

$$S_{[c(CD)][\dot{A}\ddot{A}]} = \frac{g_{1R} \left\{ (DR)(FR) + \pi \left([(CT)PR]_1 + [(CT)PR]_2 \right) \right\}}{V_{[(air)-M]} \left\{ 4\pi^2 (DR)(FR)[(CT)PR]_1 [(CT)PR]_2 + \pi (FR)^2 \left([(CT)PR]_1 + [(CT)PR]_2 \right) - (DR)(FR) - \pi \left([(CT)PR]_1 + [(CT)PR]_2 \right\} \right.} \qquad \text{(III-71)}$$

$$S_{[c(CD)][A\ddot{A}]} = \frac{g_{1R} W_{NE}}{V_{[(air)-M]} \left\{ 8\pi^2 (DR)(FR)[(CT)PR]_1 [(CT)PR]_2 + 2\pi (FR)^2 \left([(CT)PR]_1 + [(CT)PR]_2 \right) - 2(DR)(FR) - 2\pi \left([(CT)PR]_1 + [(CT)PR]_2 \right) \right\}} \qquad \text{(III-72)}$$

These are evaluated for the single case of (DR) = $\frac{1}{\sqrt{2}}$, with $[(CT)PR]_1$ = $[(CT)PR]_2$ = 0.1. Then, from (III-69), (FR) = 0.307. This gives:

$$S_{[C(CD)][\ddot{A}\dot{A}]} = -\frac{g_{IR}}{V_{[(air)-M]}\, W_{NE}\, 37.64} \qquad (III-73)$$

$$S_{[C(CD)][\dot{A}\ddot{A}]} = -\frac{g_{IR}\, 1.205}{V_{[(air)-M]}} \qquad (III-74)$$

$$S_{[C(CD)][A\ddot{A}]} = -\frac{g_{IR}\, W_{NE}\, 0.714}{V_{[(air)-M]}} \qquad (III-75)$$

Notice that once again the sensitivities are negative.

14. Plots of Results

It is now possible, using various coefficients for the equations of paragraph 11, or the indicated coefficients of the equations of paragraphs 12 and 13, to obtain numerical answers by solving the equations with various wind inputs. This was done on the Rockefeller Analyzer at Massachusetts Institute of Technology.

It is not necessary to solve for various missile speeds, since the speed of the missile can be combined with the heading angle of the missile $A_{Z(cm)}$. This is possible because of the linearization assumption which was made in deriving the kinematic equations, and

holds only if the missile airspeed is very much greater than the velocity of the wind.

Plots were made of the angular error in minutes of arc of the position of the missile perpendicular to the great circle track. Considering the earth to be spherical, this is directly a measure of distance from the track in nautical miles. For convenience, on all plots, the unit of angle is the minute of arc, and of time, the minute of time. The foot has been taken as the unit of distance,

The first input of wind which was used in solving the equations was a rectangular pulse of wind acceleration, or a constant acceleration of the velocity of the wind from zero to some finite value. The duration of the pulse was taken as one minute. Next, responses were taken for a step function input of wind acceleration. Finally, responses were determined for sinusoidal inputs of wind velocity using several frequencies. Plots of the results follow, using various sensitivities in eq (III-22):

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(DR)	$\dfrac{1}{S_{[C(CD)][\ddot{A}\ddot{A}]}\, V_{[(air)-M]}}$	$\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}$	$\dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}$
.417	0.00000208	0.4195	0.0281
.750	0.00001158	1.789	0.0885
1.050	0.00003540	4.890	0.2172
1.500	0.0001255	16.05	0.6575
1.875	0.0002891	35.75	1.4210

NUMERICAL CONSTANTS USED IN OBTAINING RESPONSE OF TRACK CONTROL SYSTEM TO PULSE AND STEP OF WIND ACCELERATION AND CIRCULATING WIND

TABLE III-1

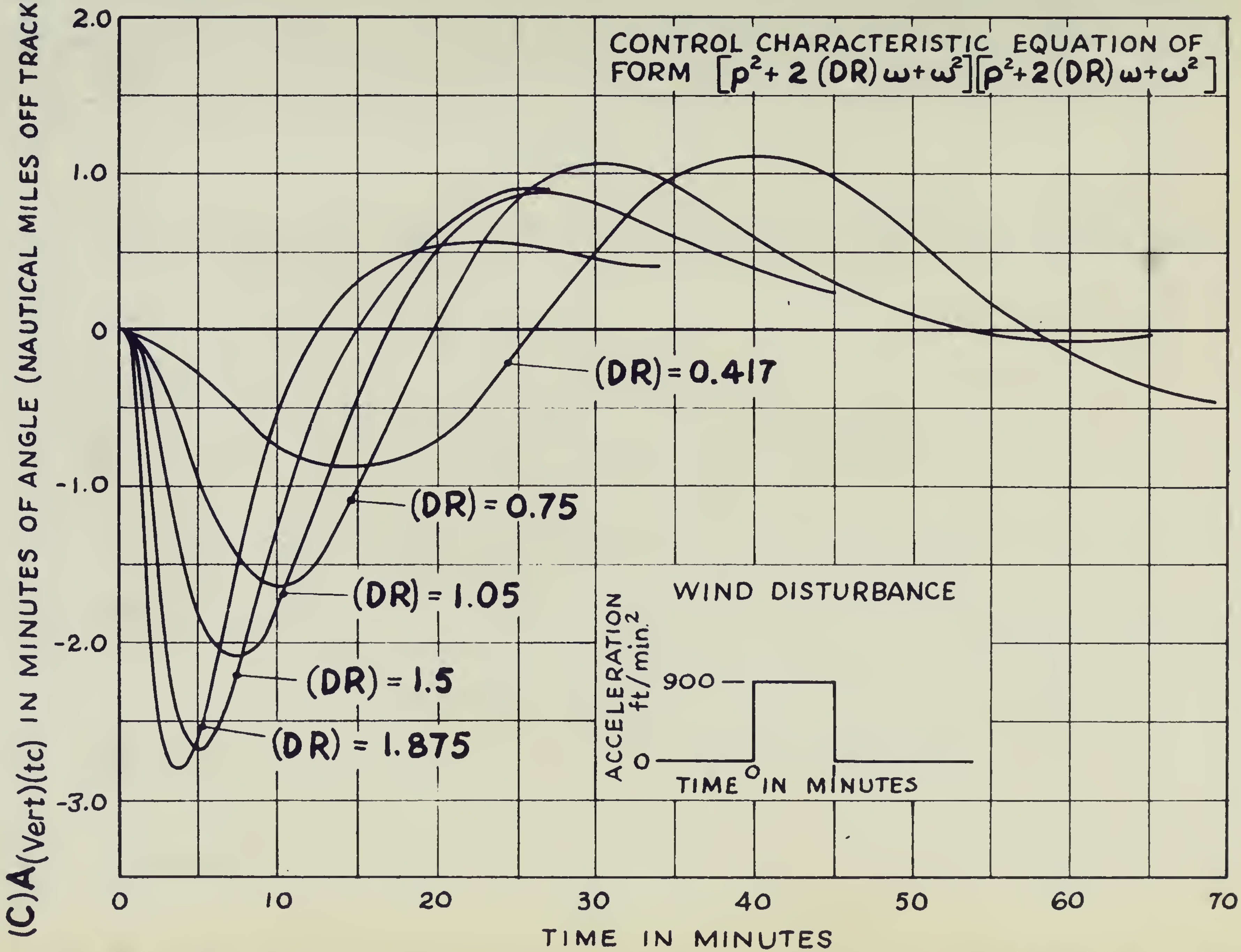

FIGURE III-2 RESPONSE OF TRACK CONTROL SYSTEM TO A PULSE DISTURBANCE OF WIND ACCELERATION

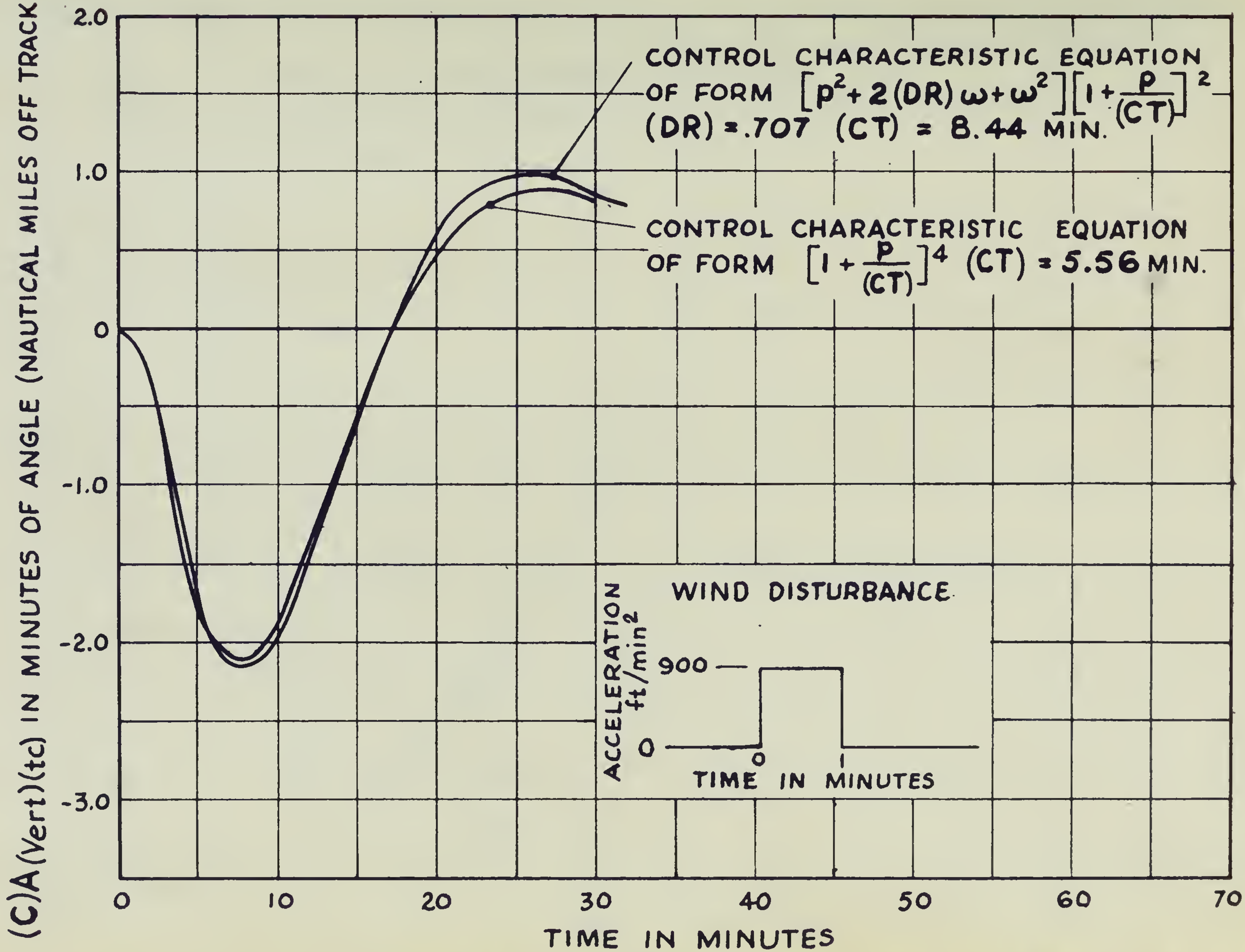

FIGURE III - 3 RESPONSE OF TRACK CONTROL SYSTEM TO A PULSE DISTURBANCE OF WIND ACCELERATION

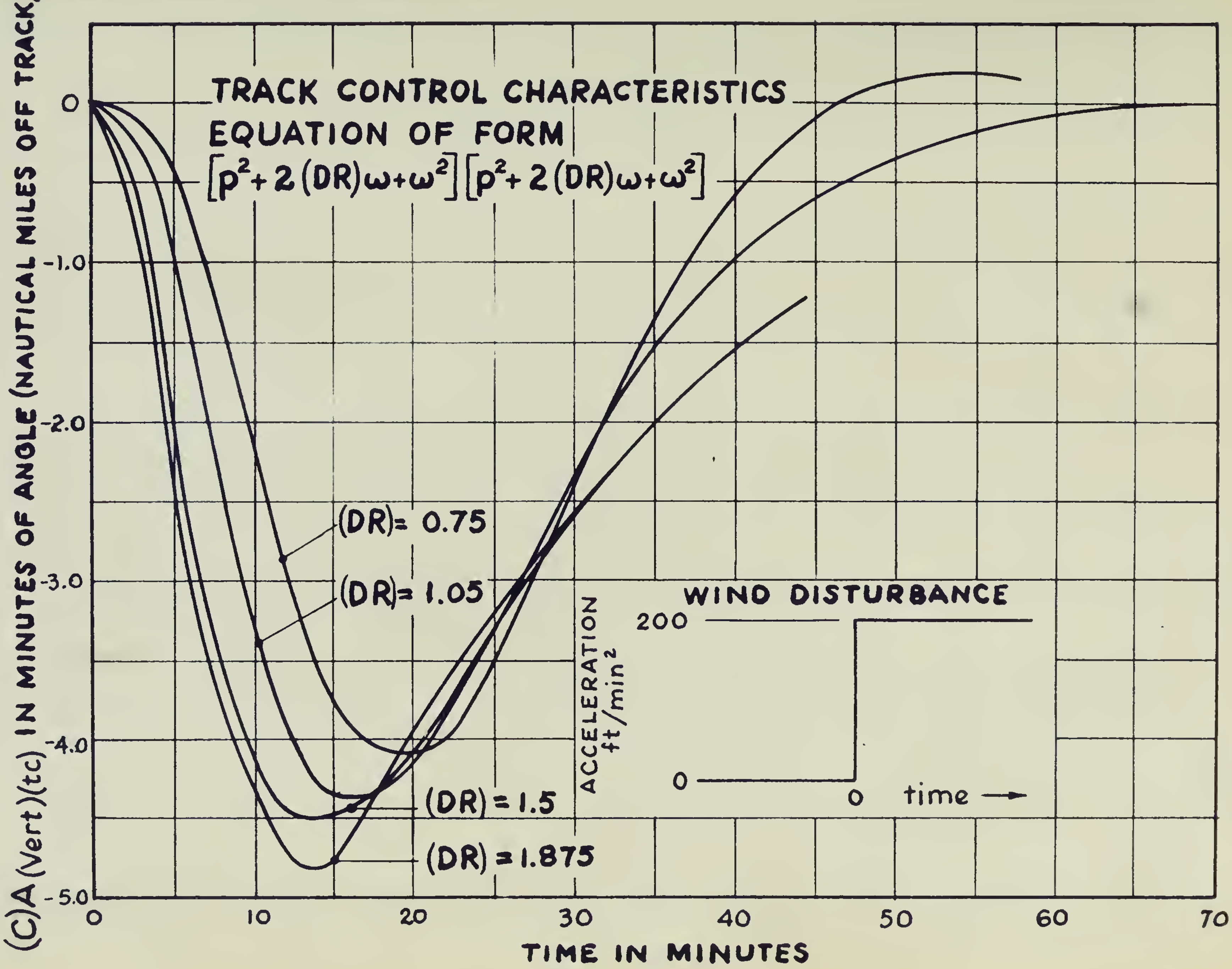

FIGURE III-4 TRANSIENT RESPONSE OF TRACK CONTROL SYSTEM TO A STEP INPUT DISTURBANCE OF WIND ACCELERATION

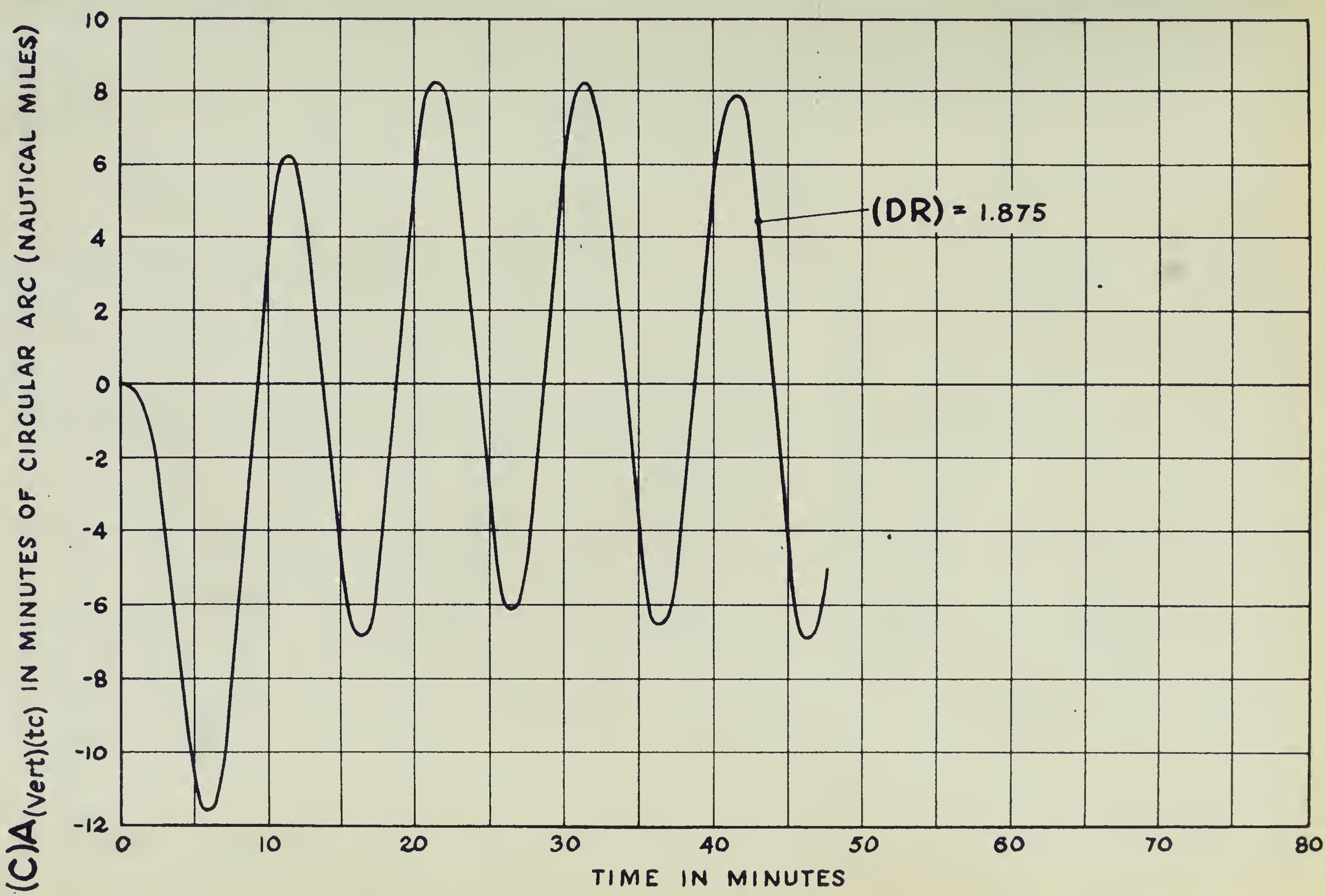

FIG. III - 5 RESPONSE OF TRACK CONTROL SYSTEM TO A 31.4 mph CIRCULATING WIND (10 MINUTE PERIOD)
(FOURTH ORDER SYSTEM)

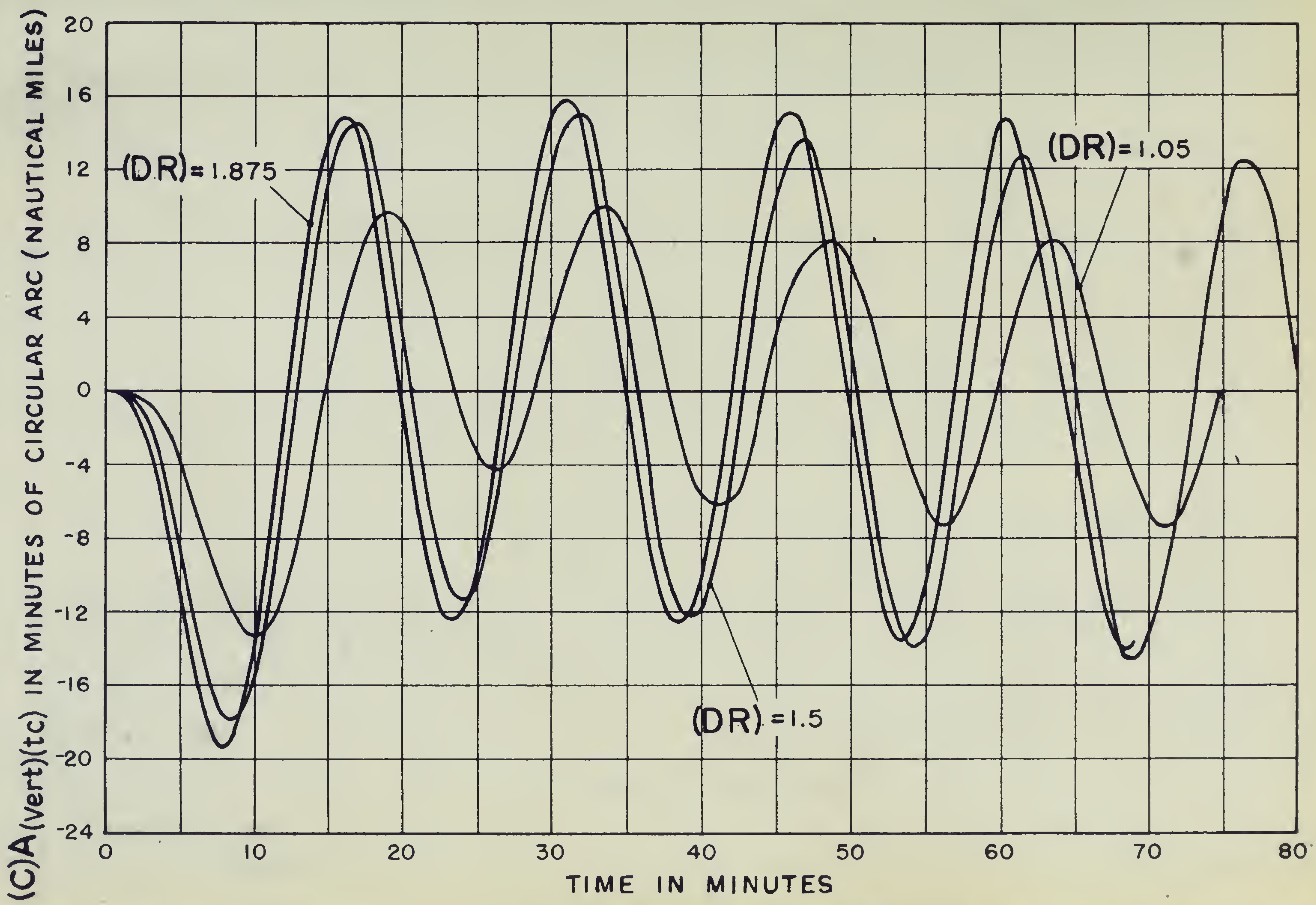

FIG. III-6 RESPONSE OF TRACK CONTROL SYSTEM TO A 31.4 mph CIRCULATING WIND (15 MINUTE PERIOD) (FOURTH ORDER SYSTEM)

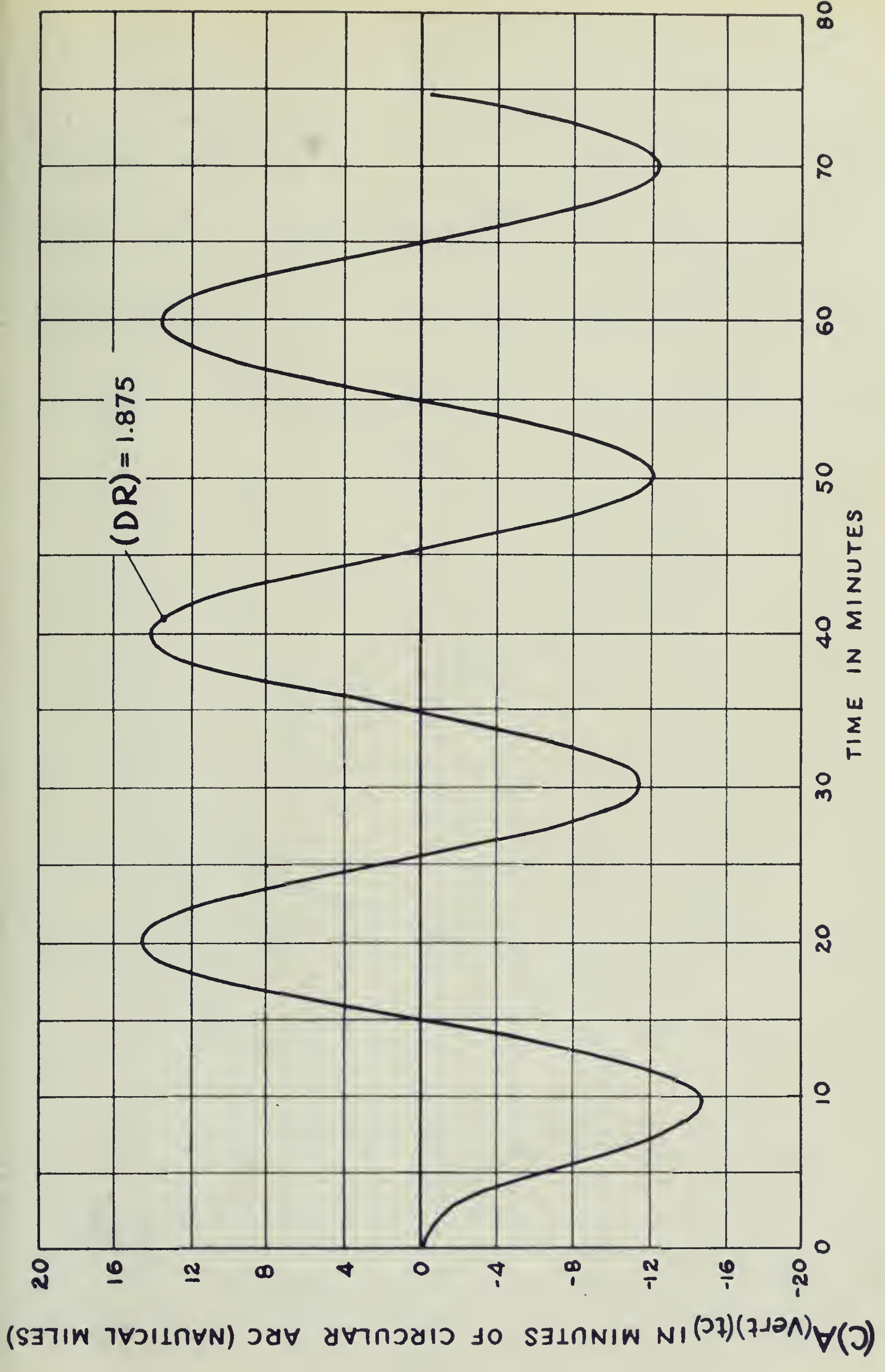

FIG. III-7 RESPONSE OF TRACK CONTROL SYSTEM TO A 31.4 mph CIRCULATING WIND (20 MINUTE PERIOD) (FOURTH ORDER SYSTEM)

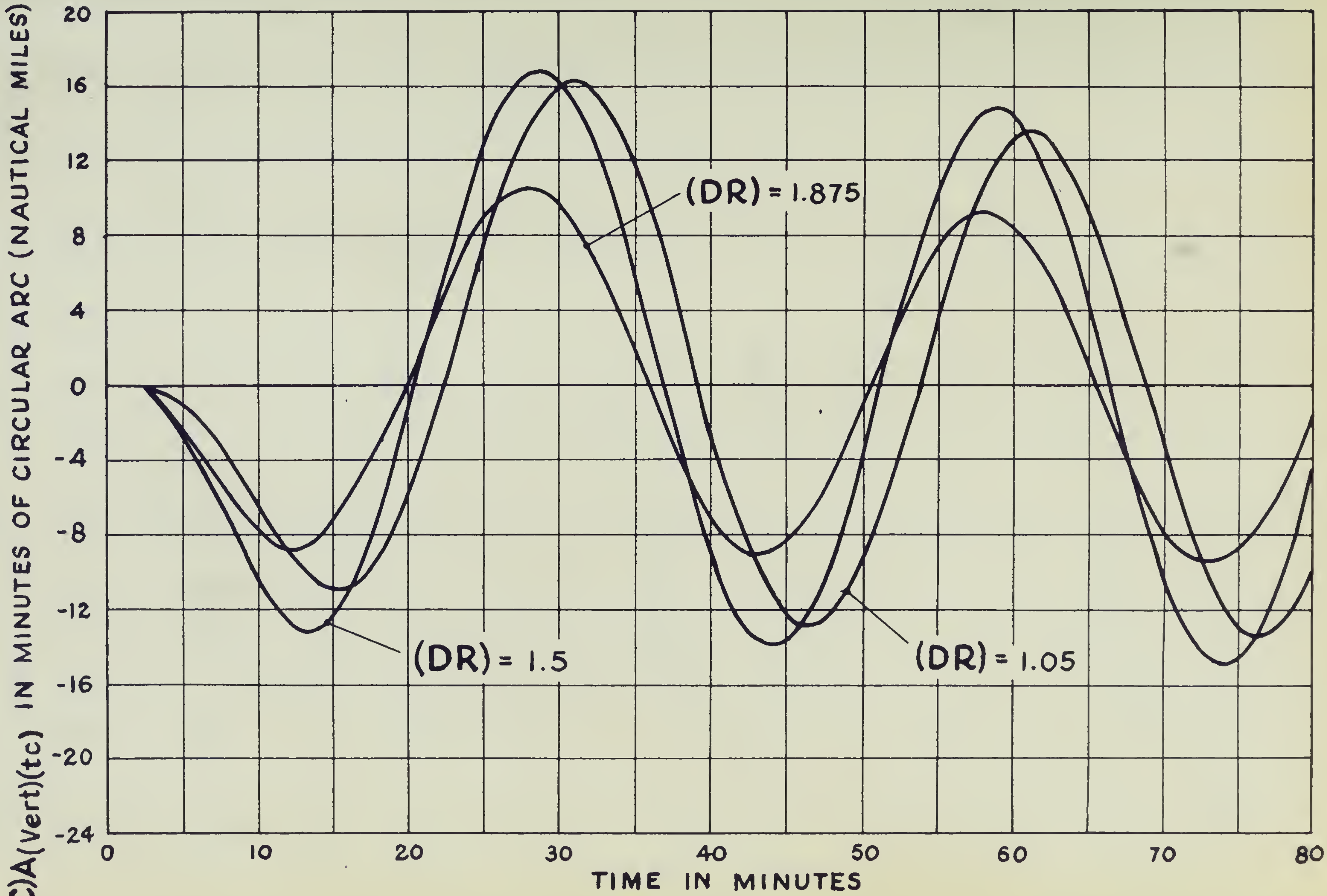

FIG. III-8 RESPONSE OF TRACK CONTROL SYSTEM TO A 31.4 mph CIRCULATING WIND (30 MINUTE PERIOD)
(FOURTH ORDER SYSTEM)

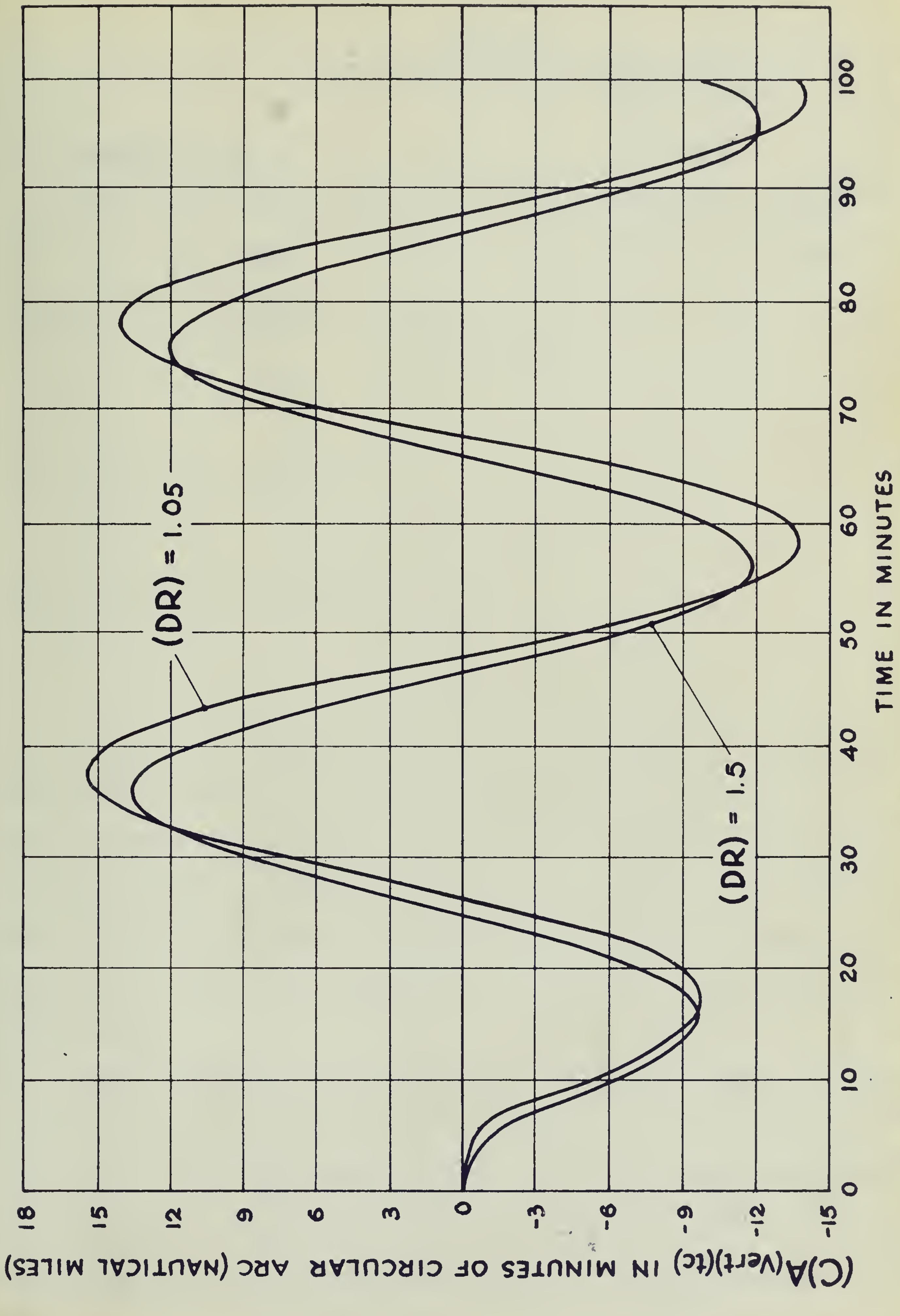

FIG. III-9 RESPONSE OF TRACK CONTROL SYSTEM TO A 31.4 mph CIRCULATING WIND (40 MINUTE PERIOD) (FOURTH ORDER SYSTEM)

CHAPTER IV

THE RANGE INDICATION PROBLEM

1. Introduction

The simplified kinematic equations developed in Chapter II for the longitudinal problem are restated here for convenience:

$$(DC)_{(ind)(long)} = \frac{\ddot{A}_{r(true)}}{W^2_{NE}} - (C)A_{(Vert)(long)} \qquad (IV\text{-}1)$$

$$A_{r(ind)} - A_{r(true)} + (C)A_r \qquad (IV\text{-}2)$$

$$A_{r(app)} = A_{r(ind)} + (DC)_{(ind)(long)} \qquad (IV\text{-}3)$$

Using these simplified kinematic equations, various mechanization equations are investigated in this chapter in an effort to determine a physically realizable and reasonably accurate mechanization of the range indication problem. The assumptions are identical with those for the lateral problem. The missile is assumed to move with instantaneous response to wind forcing functions.

The purpose of the longitudinal mechanization is to reduce toward zero any angle (C) [(Vert)(ind)(long)] which has been introduced by motion of the missile along the prescribed great circle track, and by measuring the angle between the reference vertical and the indicated vertical, to obtain a measure of the

location of the missile. The orientation of the controlled member is achieved by measuring the angle $(DC)_{(ind)(long)}$ between the position of the pendulum and the controlled member, and by operating on this quantity to position the controlled member with respect to the reference vertical. The longitudinal reference is chosen as the vertical of the departure point, $(Vert)_{(dep)}$. Since the value of $(DC)_{(ind)(long)}$ is dependent upon the position of the controlled member, as well as of the pendulum, it is more convenient, in deriving mechanizations, to use the angle $A_{r(app)}$ between the reference vertical and the indicated vertical of the controlled member, than to use $(DC)_{(ind)(long)}$ directly. $A_{r(app)}$ is physically measurable, since it is the sum of $(DC)_{(ind)(long)}$ and $A_{r(ind)}$, which are both known. It has the advantage of being independent of the position of the controlled member.

Throughout the derivations that follow, it is convenient to refer to Figure II-6, which indicates the relationships among the important angles.

2. Trial Mechanizations

A. The first mechanization that is considered is that in which the second integral of $A_{r(app)}$ moves the controlled member according to $A_{r(ind)}$ and its first two integrals. That any simpler case produces an unsatisfactory solution is indicated by the mechanization equations for the track control problem. The mechanization equation is

$$S_{[c(cm)][AA]} \iint A_{r(ind)}\, dt\, dt + S_{[c(cm)][\dot{A}A]} \int AR_{(ind)}\, dt + S_{[c(cm)][\ddot{A}A]} A_{r(ind)} = \iint A_{r(app)}\, dt\, dt \qquad (IV\text{-}4)$$

Differentiating twice,

$$S_{[c(cm)][AA]}\,A_{r(ind)}+S_{[c(cm)]\,[\dot{A}A]}\,\dot{A}_{r(ind)}+S_{[c(cm)][\ddot{A}A]}\,\ddot{A}_{r(ind)}=A_{r(app)} \tag{IV-5}$$

Substituting (IV-3) into (IV-4)

$$\left\{S_{[c(cm)][AA]}-1\right\}A_{r(ind)}+S_{[c(cm)][\dot{A}A]}\,\dot{A}_{r(ind)}+S_{[c(cm)][\ddot{A}A]}\,\ddot{A}_{r(ind)}=DC_{(ind)long} \tag{IV-6}$$

which is the mechanization equation in terms of $(DC)_{(ind)(long)}$ and $A_{r(ind)}$. Further substituting (IV-1) and (IV-2) into (IV-6),

$$-\left\{1-\frac{1}{W_{NE}^{2}\,S_{[c(cm)][\ddot{A}A]}}\right\}\ddot{A}_{r(true)}-\frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddot{A}A]}}\,\dot{A}_{r(true)}-\left\{\frac{S_{[c(cm)][AA]}}{S_{[c(cm)][\ddot{A}A]}}-\frac{1}{S_{[c(cm)][\ddot{A}A]}}\right\}A_{r(true)}=(C)\ddot{A}_{r}+\frac{S_{[c(cm)][\dot{A}A]}\,(C)\dot{A}_{r}}{S_{[c(cm)][\ddot{A}A]}}+\frac{S_{[c(cm)][AA]}\,(C)A_{r}}{S_{[c(cm)][\ddot{A}A]}} \tag{IV-7}$$

This control equation indicates that the value for $S_{[C(CM)][AA]}$ should be unity, and it will be so chosen in further considerations of the problem. Even with the position error eliminated by the proper selection of $S_{[(C)(CM)]\,[AA]}$, there remains an error caused by the rate of change of $A_{r(true)}$. Since $\frac{A_{r(true)}}{R_E}$ is a measure of the distance travelled over the surface of the earth, and because airspeed is constant,

$$\frac{\ddot{A}_{r(true)}}{R_E}=\dot{V}_{[E-(air)]} \tag{IV-8}$$

This means that the steady state error is a function of wind velocity, and that this equation is therefore unsatisfactory.

B. In order to improve this response, the first integral of $A_{r(app)}$ is added to the second integral in the preceding mechanization equation. The resulting equation is

$$\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\dot{A}A]} \int A_{r(ind)}\,dt + S_{[c(cm)][\ddot{A}A]} A_{r(ind)} = \iint A_{r(app)}\,dt\,dt + S_{[c(cm)][\dot{A}A]} \int A_{r(app)}\,dt \qquad \text{(IV-9)}$$

from which, with (IV-3),

$$S_{[c(cm)][\ddot{A}A]} \ddot{A}_{r(ind)} = (DC)_{(ind)(long)} + S_{[c(cm)][\dot{A}A]} \dot{(DC)}_{(ind)(long)} \qquad \text{(IV-10)}$$

Further substituting (IV-1) and (IV-2) into (IV-10),

$$\frac{1}{W_{NE}^2} \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddot{A}A]}} \dddot{A}_{r(true)} - \left\{ 1 - \frac{1}{W_{NE}^2} \frac{1}{S_{[c(cm)][\ddot{A}A]}} \right\} \ddot{A}_{r(true)} = (C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddot{A}A]}} (C)\dot{A}_r + \frac{1}{S_{[c(cm)][\ddot{A}A]}} (C)A_r \qquad \text{(IV-11)}$$

In this control equation, there is a forced error resulting from the acceleration of the wind which can be removed if $\frac{1}{S_{[C(CM)][\ddot{A}A]}}$ is made equal to W^2_{NE}. Under this condition, a stable control equation results with a forced error resulting from no effect of the wind of lower order than the time rate of change of acceleration. The homogeneous equation contains damping, but has a period of about 84.4 minutes, because the natural frequency of the system is fixed as W^2_{NE} when the value of $S_{[C(CM)][\ddot{A}A]}$

is chosen. Because it is desirable to increase this natural frequency, in order to reduce the time during which transient errors are of importance, other mechanization equations are examined in this chapter, in an effort to acquire control over period as well as damping in the control equation.

C. The third mechanization equation to be considered uses the third integral of $A_{r(app)}$ to move the controlled member, by operating on $A_{r(ind)}$ and its first three integrals. This mechanization can be expressed

$$\iiint A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int A_{r(ind)}\,dt + S_{[c(cm)][\dddot{A}A]}\,A_{r(ind)} = \iiint A_{r(app)}\,dt\,dt\,dt \qquad \text{(IV-12)}$$

Differentiating equation (IV-12) three times, and substituting equation (IV-3) into (IV-12),

$$S_{[c(cm)][\dot{A}A]}\,\dot{A}_{r(ind)} + S_{[c(cm)][\ddot{A}A]}\,\ddot{A}_{r(ind)} + S_{[c(cm)][\dddot{A}A]}\,\dddot{A}_{r(ind)} = (DC)_{(ind)(long)} \qquad \text{(IV-13)}$$

Further substituting equations (IV-1) and (IV-2) into eq (IV-13),

$$-\dddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{1}{W_{NE}^{2}\,S_{[c(cm)][\dddot{A}A]}}\right\}\ddot{A}_{r(true)} - \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}\,\dot{A}_{r(true)} = (C)\dddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dot{A}_r + \frac{(C)A_r}{S_{[c(cm)][\dddot{A}A]}} \qquad \text{(IV-14)}$$

This control equation again contains a forced error proportional to the velocity of the wind, and when the proportionality factor $S_{[c(cm)][\dot{A}A]}$ is made zero, the homogeneous equation becomes

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unstable.

D. In an effort to improve this situation, the second integral of $A_{r(app)}$ is added to the third integral in the preceding mechanization. Then,

$$\iiint A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int A_{r(ind)}\,dt + S_{[c(cm)][\dddot{A}A]}\,A_{r(ind)} = \iiint A_{r(app)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(app)}\,dt\,dt \tag{IV-15}$$

from which, with eq (IV-3),

$$S_{[c(cm)][\ddot{A}A]}\,\ddot{A}_{r(ind)} + S_{[c(cm)][\dddot{A}A]}\,\dddot{A}_{r(ind)} = (DC)_{(ind)\,long} + S_{[c(cm)][\dot{A}A]}\,(\dot{DC})_{(ind)\,long} \tag{IV-16}$$

Further substituting eqs (IV-1) and (IV-2) into eq (IV-16),

$$-\left\{1 - \frac{S_{[c(cm)][\dot{A}A]}}{W^2_{NE}\,S_{[c(cm)][\dddot{A}A]}}\right\}\dddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{1}{W^2_{NE}\,S_{[c(cm)][\dddot{A}A]}}\right\}\ddot{A}_{r(true)} = (C)\dddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dot{A}_r + \frac{(C)A_r}{S_{[c(cm)][\dddot{A}A]}} \tag{IV-17}$$

The lowest order of the forced error in this control equation is that of the acceleration of the wind. The coefficient of this term, however is $\left[\frac{S_{[C(CM)][\ddot{A}A]}}{S_{[C(CM)][\dddot{A}A]}} - \frac{1}{W^2_{NE}\,S_{[C(CM)][\dddot{A}A]}}\right]$. If $S_{[C(CM)][\ddot{A}A]}$ is chosen as equal to $\frac{1}{W^2_{NE}}$, the acceleration term disappears and the control equation indicates that the lowest order of forced error is that resulting from the time rate of change of the acceleration. The coefficient of this term is of similar form to that of the acceleration term, and is

$\left[1 - \frac{S_{[C(CM)]\,[\dot{A}A]}}{W^2_{NE}\, S_{[C(CM)]\,[\ddot{A}A]}}\right]$. It thus appears that the acceleration rate term can be made to disappear if $\frac{S_{[C(CM)][\dot{A}A]}}{S_{[C(CM)][\ddot{A}A]}}$ is chosen as W^2_{NE}.

From Routh's stability criteria for a cubic, the sign of each term in the homogeneous equation must be positive, and

$$\left[\frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddot{A}A]}}\right]\left[\frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}\right] > \frac{1}{S_{[c(cm)][\ddot{A}A]}} \qquad \text{(IV-18)}$$

Equation (IV-18) shows that, with $S_{[C(CM)][\ddot{A}A]} = \frac{1}{W^2_{NE}}$, the system is stable only if $\frac{S_{[C(CM)]\,[\dot{A}A]}}{S_{[C(CM)]\,[\dddot{A}A]}}$ is greater that W^2_{NE}. The acceleration rate term can therefore not be eliminated if the system is to remain stable. The equation that has been considered here, if it is physically possible to mechanize it, appears to provide a satisfactory system. It remains to be seen whether additional terms in the mechanization equation can improve the response.

E. Consider next, therefore, the performance which results when to the mechanization equation just considered, the first integral of $A_{r(app)}$ is added to the second and third. This mechanization can be written

$$\iiint A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int A_{r(ind)}\,dt + S_{[c(cm)][\dddot{A}A]}A_{r(ind)} = \iiint A_{r(app)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(app)}\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int A_{r(app)}\,dt \qquad \text{(IV-19)}$$

from which, with eq (IV-3),

$$S_{[c(cm)][\dddot{A}A]}\,\ddot{A}_{r(ind)} = (DC)_{(ind)\,long} + S_{[c(cm)][\dot{A}A]}\,(\dot{DC})_{(ind)\,long} + S_{[c(cm)][\ddot{A}A]}\,(\ddot{DC})_{(ind)\,long} \tag{IV-20}$$

Further substituting eqs (IV-1) and (IV-2) into eq (IV-20),

$$\frac{1}{W^2_{NE}}\,\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}\,\ddddot{A}_{r(true)} - \left\{1 - \frac{1}{W^2_{NE}}\,\frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}\right\}\dddot{A}_{r(true)} + \frac{1}{W^2_{NE}}\,\frac{1}{S_{[c(cm)][\dddot{A}A]}}\,\ddot{A}_{r(true)} = (C)\dddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dot{A}_r + \frac{1}{S_{[c(cm)][\dddot{A}A]}}(C)A_r \tag{IV-21}$$

Here again, the lowest order forced-error term contains the acceleration of the wind. The coefficient of this term is now $\frac{1}{W^2_{NE}}$, and does not contain any controllable sensitivity. This system is therefore less satisfactory than that of par 2-D.

F. For the sake of completeness, $A_{r(app)}$ is next added to its integrals in the mechanization equation just discussed. This gives

$$\iiint A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int A_{r(ind)}\,dt + S_{[c(cm)][\dddot{A}A]}\,A_{r(ind)} = \iiint A_{r(app)}\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iint A_{r(app)}\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int A_{r(app)}\,dt + S_{[c(cm)][\dddot{A}A]}\,A_{r(app)} \tag{IV-22}$$

from which, with eq (IV-3),

$$(DC)_{(ind)\,long} + S_{[c(cm)][\dot{A}A]}\,(\dot{DC})_{(ind)\,long} + S_{[c(cm)][\ddot{A}A]}\,(\ddot{DC})_{(ind)\,long} + S_{[c(cm)][\dddot{A}A]}\,(\dddot{DC})_{(ind)\,long} = 0 \tag{IV-23}$$

Further substituting eqs (IV-1) and (IV-2) into eq (IV-23)

$$\frac{1}{W_{NE}^2}\overset{\cdot\cdot\cdot\cdot\cdot}{\ddot{A}}_{r(true)}+\frac{1}{W_{NE}^2}\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\ddot{A}A]}}\overset{\cdot\cdot\cdot}{\ddot{A}}_{r(true)}+\frac{1\,S_{[c(cm)][\dot{A}A]}}{W_{NE}^2 S_{[c(cm)][\ddot{A}A]}}\overset{\cdot\cdot\cdot}{\ddot{A}}_{r(true)}-\frac{1}{W_{NE}^2 S_{[c(cm)][\ddot{A}A]}}\ddot{A}_{r(true)}=(C)\overset{\cdot\cdot\cdot}{\ddot{A}}_r+\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\ddot{A}A]}}(C)\ddot{A}_r+\frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddot{A}A]}}(C)\dot{A}_r+\frac{1}{S_{[c(cm)][\ddot{A}A]}}(C)A_r \qquad \text{(IV-24)}$$

This system, like that of par 2-E, contains an unremovable steady-state error proportional to the magnitude of the wind acceleration.

G. When examining possible mechanization equations, it next seems reasonable to add to the system of par 2-D the fourth integral of $A_{r(ind)}$, using the fourth integral of $A_{r(app)}$. This gives

$$\iiiint A_{r(ind)}\,dt\,dt\,dt\,dt+S_{[c(cm)][\dot{A}A]}\iiint A_{r(ind)}\,dt\,dt\,dt+S_{[c(cm)][\ddot{A}A]}\iint A_{r(ind)}\,dt\,dt+S_{[c(cm)][\dddot{A}A]}\int A_{r(ind)}\,dt+S_{[c(cm)][\ddddot{A}A]}A_{r(ind)}=\iiiint A_{r(app)}\,dt\,dt\,dt\,dt \qquad \text{(IV-25)}$$

from which, with (IV-3),

$$S_{[c(cm)][\dot{A}A]}\dot{A}_{r(ind)}+S_{[c(cm)][\ddot{A}A]}\ddot{A}_{r(ind)}+S_{[c(cm)][\dddot{A}A]}\dddot{A}_{r(ind)}+S_{[c(cm)][\ddddot{A}A]}\ddddot{A}_{r(ind)}=(DC)_{(ind)\,long} \qquad \text{(IV-26)}$$

Further substituting eqs (IV-1) and (IV-2) into (IV-26),

$$-\ddddot{A}_{r(true)} - \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}\dddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{1}{W^2_{NE}\,S_{[c(cm)][\dddot{A}A]}}\right\}\ddot{A}_{r(true)} - \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}\dot{A}_{r(true)} = (C)\ddddot{A}_{r(true)} + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dot{A}_r + \frac{1}{S_{[c(cm)][\dddot{A}A]}}(C)A_r \qquad \text{(IV-27)}$$

As was true with the corresponding system of par 2-C, an error caused by the velocity of the wind results.

H. When the third integral of $A_{r(app)}$ is added to the fourth integral used in par 2-G, the equation becomes

$$\iiiint A_{r(ind)}\,dt\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iiint A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\dddot{A}A]}\int A_{r(ind)}\,dt + S_{[c(cm)][\ddddot{A}A]}A_{r(ind)} = \iiiint A_{r(app)}\,dt\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iiint A_{r(app)}\,dt\,dt\,dt \qquad \text{(IV-28)}$$

from which, with eq (IV-3)

$$S_{[c(cm)][\dot{A}A]}\ddot{A}_{r(ind)} + S_{[c(cm)][\ddot{A}A]}\dddot{A}_{r(ind)} + S_{[c(cm)][\dddot{A}A]}\ddddot{A}_{r(ind)} = (DC)_{(ind)\,long} + S_{[c(cm)][\dot{A}A]}(\dot{DC})_{(ind)\,long} \qquad \text{(IV-29)}$$

Further substituting eq (IV-1) into eq (IV-29),

$$-\ddddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{S_{[c(cm)][\dot{A}A]}}{W^2_{NE}\,S_{[c(cm)][\dddot{A}A]}}\right\}\dddot{A}_{(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{1}{W^2_{NE}\,S_{[c(cm)][\dddot{A}A]}}\right\}\ddot{A}_{r(true)} = (C)\ddddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}}(C)\dot{A}_r + \frac{1}{S_{[c(cm)][\dddot{A}A]}}(C)A_r \qquad \text{(IV-30)}$$

Here, as with the system of par 2-D, the control equation indicates that the lowest order of error in the steady state results from the acceleration of the wind. The coefficient of this term is $\frac{S_{[C(CM)][\ddot{A}A]}}{S_{[C(CM)][\dddot{A}A]}} - \frac{1}{W^2_{NE}\,S_{[C(CM)][\dddot{A}A]}}$.

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As before, if $S_{[C(CM)][\ddot{AA}]}$ is made equal to $\frac{1}{W^2_{NE}}$, the acceleration term disappears. The coefficient of the acceleration rate term is $\frac{S_{[C(CM)][\dddot{AA}]}}{S_{[C(CM)][\ddddot{AA}]}} - \frac{S_{[C(CM)][\dot{AA}]}}{W^2_{NE}\, S_{[C(CM)][\ddddot{AA}]}}$. It appears that this term can be made to disappear if $\frac{S_{[C(CM)][\dddot{AA}]}}{S_{[C(CM)][\dot{AA}]}}$ is chosen equal to $\frac{1}{W^2_{NE}}$.

From Routh's stability criteria for a quartic, the sign of each term in the homogeneous equation must be positive, and

$$S_{[c(cm)][\dddot{AA}]}\, S_{[c(cm)][\ddot{AA}]}\, S_{[c(cm)][\dot{AA}]} - S_{[c(cm)][\ddddot{AA}]}\left\{S_{[c(cm)][\dot{AA}]}\right\}^2 - \left\{S_{[c(cm)][\dddot{AA}]}\right\}^2 > 0 \qquad \text{(IV-31)}$$

If $S_{[C(CM)][\ddot{AA}]}$ is made equal to $\frac{1}{W^2_{NE}}$, eq (IV-31) becomes

$$\left\{\frac{S_{[c(cm)][\dddot{AA}]}}{S_{[c(cm)][\dot{AA}]}}\right\}\left\{\frac{S_{[c(cm)][\dot{AA}]}}{W_{NE}}\right\}^2 - S_{[c(cm)][\ddddot{AA}]}\left\{S_{[c(cm)][\dot{AA}]}\right\}^2 - \left\{\frac{S_{[c(cm)][\dddot{AA}]}}{S_{[c(cm)][\dot{AA}]}}\right\}^2\left\{S_{[c(cm)][\dot{AA}]}\right\}^2 > 0 \qquad \text{(IV-32)}$$

Therefore, in order to eliminate the acceleration rate term, $- S_{[C(CD)][\ddddot{AA}]}\left\{S_{[C(CD)][\dot{AA}]}\right\}^2$ must be greater than zero, which is possible only if $S_{[C(CD)][\ddddot{AA}]}$ becomes negative. Routh's first criterion of stability, which forbids variation in the signs of the quartic coefficient, is then violated, and the system is unstable.

Once again, therefore, it has not been possible to eliminate, in the steady state, any effect of wind of higher order than acceleration. Mechanization eq (IV-28) appears to give the same accuracy, in the steady state, that results from the use of eq (IV-15). In choosing between these equations, it is necessary to compare dynamic response and ease of mechanization. Before an examination is made of this problem, a study will be made of other mechanization equations.

I. In an effort to eliminate the effect of acceleration rate, the second integral of $A_{r(app)}$ is next added to the mechanization equation of par 2-G. This gives

$$\iiiint A_{r(ind)}\,dt\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iiint A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\iint A_{r(ind)}\,dt\,dt + S_{[c(cm)][\dddot{A}A]}\int A_{r(ind)}\,dt\,dt + S_{[c(cm)][\ddddot{A}A]}A_{r(ind)} = \iiiint A_{r(app)}\,dt\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\iiint A_{r(app)}\,dt\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\iint A_{r(app)}\,dt\,dt \tag{IV-33}$$

from which, with eq (IV-3),

$$S_{[c(cm)][\ddot{A}A]}\dddot{A}_{r(ind)} + S_{[c(cm)][\dddot{A}A]}\ddddot{A}_{r(ind)} = (DC)_{(ind)long} + S_{[c(cm)][\dot{A}A]}(\dot{D}C)_{(ind)long} + S_{[c(cm)][\ddot{A}A]}(\ddot{D}C)_{(ind)long} \tag{IV-34}$$

Further substituting eqs (IV-1) and (IV-2) into eq (IV-34)

$$-\left\{1-\frac{S_{[c(cm)][\ddot{A}A]}}{W_{NE}^2\,S_{[c(cm)][\ddddot{A}A]}}\right\}\dddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\ddddot{A}A]}} - \frac{S_{[c(cm)][\dot{A}A]}}{W_{NE}^2\,S_{[c(cm)][\ddddot{A}A]}}\right\}\ddot{A}_{r(true)} + \frac{1}{W_{NE}^2\,S_{[c(cm)][\ddddot{A}A]}}\ddot{A}_{r(true)} = (C)\ddddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\ddddot{A}A]}}(C)\dddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\ddddot{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddddot{A}A]}}(C)\dot{A}_r + \frac{1}{S_{[c(cm)][\ddddot{A}A]}}(C)A_r \tag{IV-35}$$

This control equation shows that, instead of improving the response, the use of eq (IV-33) has resulted in an acceleration error which it is no longer possible to eliminate.

J. A final effort to remove the acceleration rate term is made by adding the fifth integral of $A_{r(app)}$ and of $A_{r(ind)}$ to the mechanization equation used in par 2-G. This gives,

$$\int\!\!\int\!\!\int\!\!\int\!\!\int A_{r(ind)}\,dt\,dt\,dt\,dt\,dt - S_{[c(cm)][\dot{A}A]}\int\!\!\int\!\!\int\!\!\int A_{r(ind)}\,dt\,dt\,dt\,dt + S_{[c(cm)][\ddot{A}A]}\int\!\!\int\!\!\int A_{r(ind)}\,dt\,dt\,dt + S_{[c(cm)][\dddot{A}A]}\int\!\!\int A_{r(ind)}\,dt\,dt + S_{[c(cm)][\ddddot{A}A]}\int A_{r(ind)}\,dt + S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}\,A_{r(ind)} = \int\!\!\int\!\!\int\!\!\int\!\!\int A_{r(app)}\,dt\,dt\,dt\,dt\,dt + S_{[c(cm)][\dot{A}A]}\int\!\!\int\!\!\int\!\!\int A_{r(app)}\,dt\,dt\,dt\,dt \tag{IV-36}$$

from which, with eq (IV-3),

$$S_{[c(cm)][\ddot{A}A]}\ddot{A}_{r(ind)} + S_{[c(cm)][\dddot{A}A]}\dddot{A}_{r(ind)} + S_{[c(cm)][\ddddot{A}A]}\ddddot{A}_{r(ind)} + S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}\overset{\cdot\cdot\cdot\cdot\cdot}{A}_{r(ind)} = (DC)_{(ind)\,long} + S_{[c(cm)][\dot{A}A]}\,(\dot{DC})_{(ind)\,long} \tag{IV-37}$$

Further substituting eqs (IV-1) and (IV-2) into eq (IV-37),

$$-\overset{\cdot\cdot\cdot\cdot\cdot}{A}_{r(true)} - \frac{S_{[c(cm)][\ddddot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}\ddddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\dddot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}} - \frac{S_{[c(cm)][\dot{A}A]}}{W^2_{NE}\,S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}\right\}\dddot{A}_{r(true)} - \left\{\frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}} - \frac{1}{W^2_{NE}\,S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}\right\}\ddot{A}_{r(true)} = (C)\overset{\cdot\cdot\cdot\cdot\cdot}{A}_r + \frac{S_{[c(cm)][\ddddot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}(C)\ddddot{A}_r + \frac{S_{[c(cm)][\dddot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}(C)\dddot{A}_r + \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}(C)\ddot{A}_r + \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}(C)\dot{A}_r + \frac{1}{S_{[c(cm)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}(C)A_r \tag{IV-38}$$

The coefficient of the wind acceleration term of the forced error now has the form $\left[\frac{S_{[C(CM)][\ddot{A}A]}}{S_{[C(CM)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}} - \frac{1}{W^2_{NE}\,S_{[C(CM)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}\right]$.

To eliminate this, $S_{[C(CM)][\ddot{A}A]}$ must be made equal to $\frac{1}{W^2_{NE}}$, as before. In this equation, the wind acceleration term has the coefficient $\left[\frac{S_{[C(CM)][\dddot{A}A]}}{S_{[C(CM)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}} - \frac{S_{[C(CM)][\dot{A}A]}}{W^2_{NE}\,S_{[C(CM)][\overset{\cdot\cdot\cdot\cdot\cdot}{A}A]}}\right]$. If this is to be eliminated, $\left[\frac{S_{[C(CM)][\dddot{A}A]}}{S_{[C(CM)][\dot{A}A]}}\right]$ must be made equal to $\frac{1}{W^2_{NE}}$.

Using Routh's stability criteria for a quintic, the coefficients of the homogeneous equation must all be positive, and

$$S_{[c(cm)][\dot{A}A]} - \frac{S_{[c(cm)][\ddddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{\left\{S_{[c(cm)][\ddddot{A}A]}\, S_{[c(cm)][\dddot{A}A]} - S_{[c(cm)][\overset{.....}{A}A]}\, S_{[c(cm)][\ddot{A}A]}\right\}^2}{S_{[c(cm)][\ddddot{A}A]}\, S_{[c(cm)][\ddot{A}A]}\left\{S_{[c(cm)][\ddddot{A}A]}\, S_{[c(cm)][\dddot{A}A]} - S_{[c(cm)][\overset{.....}{A}A]}\, S_{[c(cm)][\ddot{A}A]}\right\} - \left\{S_{[c(cm)][\ddddot{A}A]}\right\}^2 \left\{S_{[c(cm)][\ddddot{A}A]}\, S_{[c(cm)][\dot{A}A]} - S_{[c(cm)][\overset{.....}{A}A]}\right\}} > 0 \tag{IV-39}$$

The system is stable, with proper choice of the other coefficients, if $S_{[C(CM)][\ddot{A}A]}$ is made equal to $\frac{1}{W^2_{NE}}$, eliminating the acceleration term. If this substitution is made in eq (IV-39) and if in addition the substitution $\frac{S_{[C(CM)][\dddot{A}A]}}{S_{[C(CM)][\dot{A}A]}} = \frac{1}{W^2_{NE}}$ is made,

$$S_{[c(cm)][\dot{A}A]} - \frac{S_{[c(cm)][\ddddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} - \frac{\left\{S_{[c(cm)][\ddddot{A}A]}\, \frac{S_{[c(cm)][\dot{A}A]}}{W^2_{NE}} - \frac{S_{[c(cm)][\overset{.....}{A}A]}}{W^2_{NE}}\right\}^2}{\frac{S_{[c(cm)][\ddddot{A}A]}}{W^2_{NE}}\left\{S_{[c(cm)][\ddddot{A}A]}\, \frac{S_{[c(cm)][\dot{A}A]}}{W^2_{NE}} - \frac{S_{[c(cm)][\overset{.....}{A}A]}}{W^2_{NE}}\right\} - \left\{S_{[c(cm)][\ddddot{A}A]}\right\}^2 \left\{S_{[c(cm)][\ddddot{A}A]}\, S_{[c(cm)][\dot{A}A]} - S_{[c(cm)][\overset{.....}{A}A]}\right\}} > 0 \tag{IV-40}$$

This equation simplifies to

$$S_{[c(cm)][\dot{A}A]} - \frac{S_{[c(cm)][\dddot{A}A]}}{S_{[c(cm)][\ddot{A}A]}} - \frac{\left(\frac{1}{W_{NE}^2}\right)^2 \left\{ S_{[c(cm)][\ddot{A}A]} \, S_{[c(cm)][\dot{A}A]} - S_{[c(cm)][\dddot{A}A]} \right\}}{\left(\frac{1}{W_{NE}^2}\right)^2 \left\{ S_{[c(cm)][\ddot{A}A]} \right\} - \left\{ S_{[c(cm)][\ddddot{A}A]} \right\}} > 0 \qquad \text{(IV-41)}$$

Since all of the sensitivities are positive, this equation can be divided by $\left[\frac{\left(\frac{1}{W^2_{NE}}\right)^2 - S_{[C(CM)][\dddot{A}A]}}{S_{[C(CM)][\ddddot{A}A]}} \right]$, provided that only $S_{[C(CD)][\dddot{A}A]}$ is maintained smaller than $\left(\frac{1}{W^2_{NE}}\right)^2$, without changing the inequality sign. This yields

$$\left\{ S_{[c(cm)][\ddot{A}A]} \, S_{[c(cm)][\dot{A}A]} - S_{[c(cm)][\dddot{A}A]} \right\} \left\{ \left(\frac{1}{W_{NE}^2}\right)^2 - S_{[c(cm)][\ddot{A}A]} \right\} - \left(\frac{1}{W_{NE}^2}\right)^2 \left\{ S_{[c(cm)][\ddot{A}A]} \, S_{[c(cm)][\dot{A}A]} - S_{[c(cm)][\ddddot{A}A]} \right\} > 0 \qquad \text{(IV-42)}$$

or, simplifying,

$$-S_{[c(cm)][\ddddot{A}A]} > 0 \qquad \text{(IV-43)}$$

Since $S_{[C(CM)][\dddot{A}A]}$ must be kept positive in order to satisfy the first Routh criterion, both criteria cannot simultaneously be satisfied, and the system becomes unstable if an attempt is made to eliminate the acceleration rate steady-state error.

3. Implementation of Equations

A. Four mechanization equations have now been discovered that eliminate steady-state error caused by the acceleration of the wind. It has not been possible to eliminate any higher order terms, but is seems unlikely that any steady-state error resulting from the acceleration rate of the wind can have appreciable effect. The simplest of these equations was eliminated because natural frequency could not be controlled. The three usable equations are (IV-15), (IV-28), and (IV-36). In eq (IV-15), two of the three sensitivities can be controlled; in eq (IV-28), three of four sensitivities can be controlled; and in eq (IV-36), four of five sensitivities can be controlled. In order to obtain the most satisfactory possible dynamic response, the use of one of the more complicated equations may be indicated, but this choice of equations is subordinate to the considerations of ease and accuracy of mechanization. A practicable method of implementing eqs (IV-15) and (IV-28) is now presented, and the complexities compared.

B. In mechanizing eq (IV-15), $(DC)_{(ind)(long)}$ is used instead of $A_{r(app)}$ as the output of the pendulous accelerometer, as explained in par 1 of this chapter. The resulting eq (IV-16) is restated here for convenience, in integrated form:

$$S_{[c(cm)][\ddot{A}A]} \int A_{r(ind)}\, dt + S_{[c(cm)][\ddot{A}A]} A_{r(ind)} = \iiint (DC)_{(ind)long}\, dt\, dt\, dt + S_{[c(cm)][\dot{A}A]} \iint (DC)_{(ind)long}\, dt\, dt \qquad \text{(IV-44)}$$

Now, an imaginary auxiliary direction is established, as shown in Fig IV-1. This direction is controlled, with respect to the reference vertical, by the equation

$$A_{(aux)(ref)} = S_{[c(aux)][A\ddot{A}]} \iint (DC)_{(ind)\,long}\, dt\, dt \tag{IV-45}$$

The controlled member is positioned from the reference vertical through the use of the auxiliary direction by the equation

$$A_{r(ind)} = -S_{[c(cm)(aux)][A\ddot{A}]} \iint (DC)_{(ind)\,long}\, dt\, dt - S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int A_{r(ind)(aux)}\, dt \tag{IV-46}$$

These equations, when combined with the relation

$$A_{r(ind)} = A_{(aux)(ref)} + A_{(ind)(aux)} \tag{IV-47}$$

become

$$A_{r(ind)} + S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int A_{r(ind)}\, dt = S_{[c(cm)(aux)][A\ddot{A}]} \iint (DC)_{(ind)\,long}\, dt\, dt + S_{[c(cm)(aux)][\dot{A}\ddot{A}]}\, S_{[c(aux)][A\ddot{A}]} \iiint (DC)_{(ind)\,long}\, dt\, dt \tag{IV-48}$$

This is identical with the desired eq (IV-44), with the relationships among sensitivities that

$$S_{[c(cm)(aux)][\dot{A}\ddot{A}]} = \frac{S_{[c(cm)][\ddot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} \tag{IV-49}$$

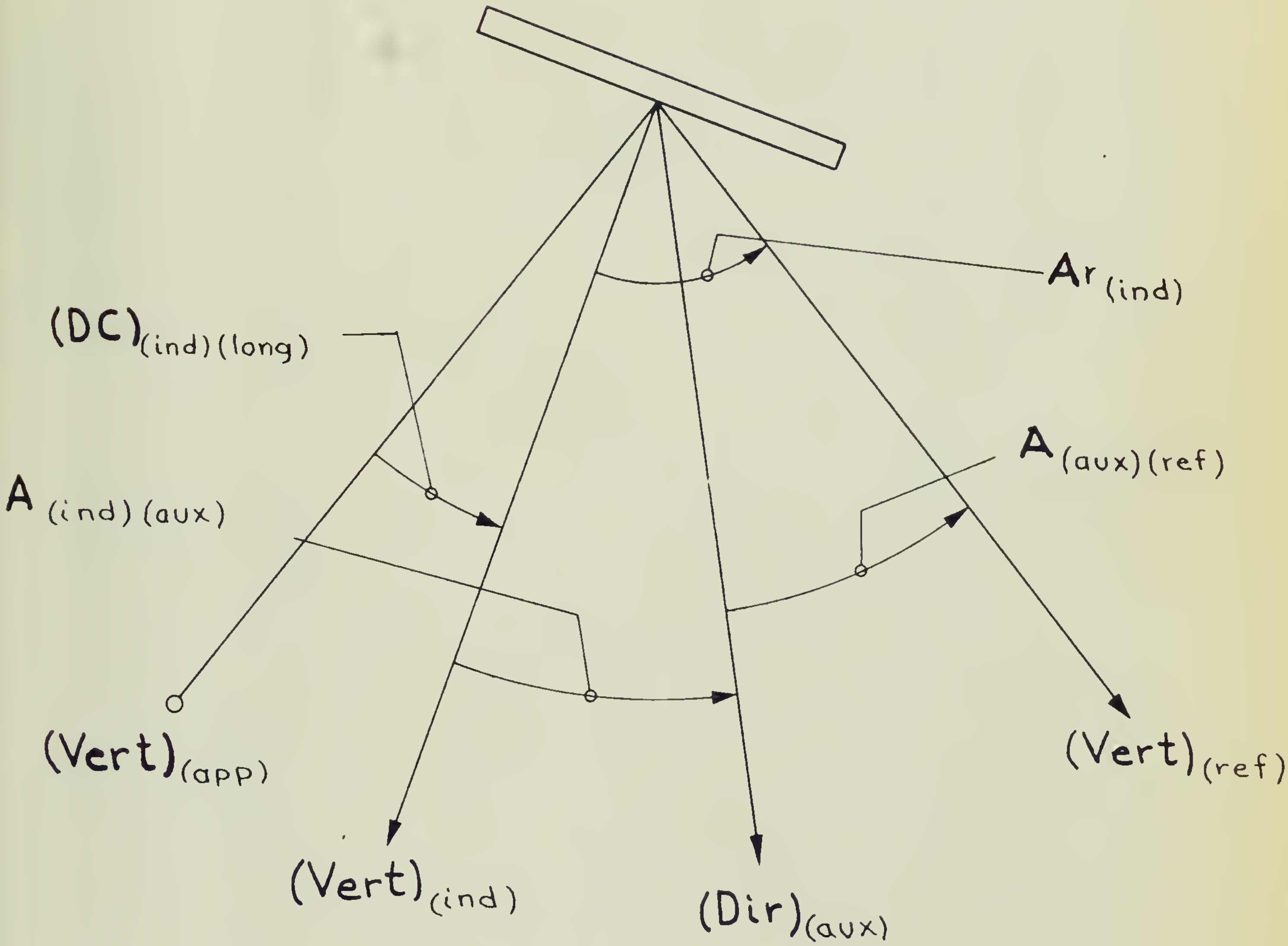


FIG. IV – 1
AUXILIARY DIRECTION FOR CUBIC MECHANIZATION EQUATION

$$S_{[c(cm)(aux)][A\ddot{A}]} = \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\dddot{A}A]}} \tag{IV-50}$$

$$S_{[c(aux)][A\ddot{A}]} = \frac{1}{S_{[c(cm)][\ddot{A}A]}} \tag{IV-51}$$

Equations (IV-45) and (IV-46) can be mechanized easily, as shown in Fig IV-2. The mechanization of eq (IV-15) is therefore feasible.

C. Equation (IV-28) is now to be mechanized in a manner similar to that used for eq (IV-15). The integrated form of the equation, using $(DC)_{(ind)(long)}$ instead of $A_{r(app)}$, as before becomes

$$S_{[c(cm)][\dddot{A}A]} A_{r(ind)} + S_{[c(cm)][\dddot{A}A]} \int A_{r(ind)}\, dt + S_{[c(cm)][\ddot{A}A]} \iint A_{r(ind)}\, dt\, dt = S_{[c(cm)][\dot{A}A]} \iiint (DC)_{(ind)long}\, dt\, dt\, dt + \iiiint (DC)_{(ind)long}\, dt\, dt\, dt\, dt \tag{IV-52}$$

For this equation, a pair of auxiliary directions is established, as shown in Fig IV-3. These directions are controlled, with respect to the reference vertical, by the equations

$$\dot{A}_{(aux)(ref)a} = S_{[c(aux)a][A\ddot{A}]} \iint (DC)_{(ind)long}\, dt\, dt \tag{IV-53}$$

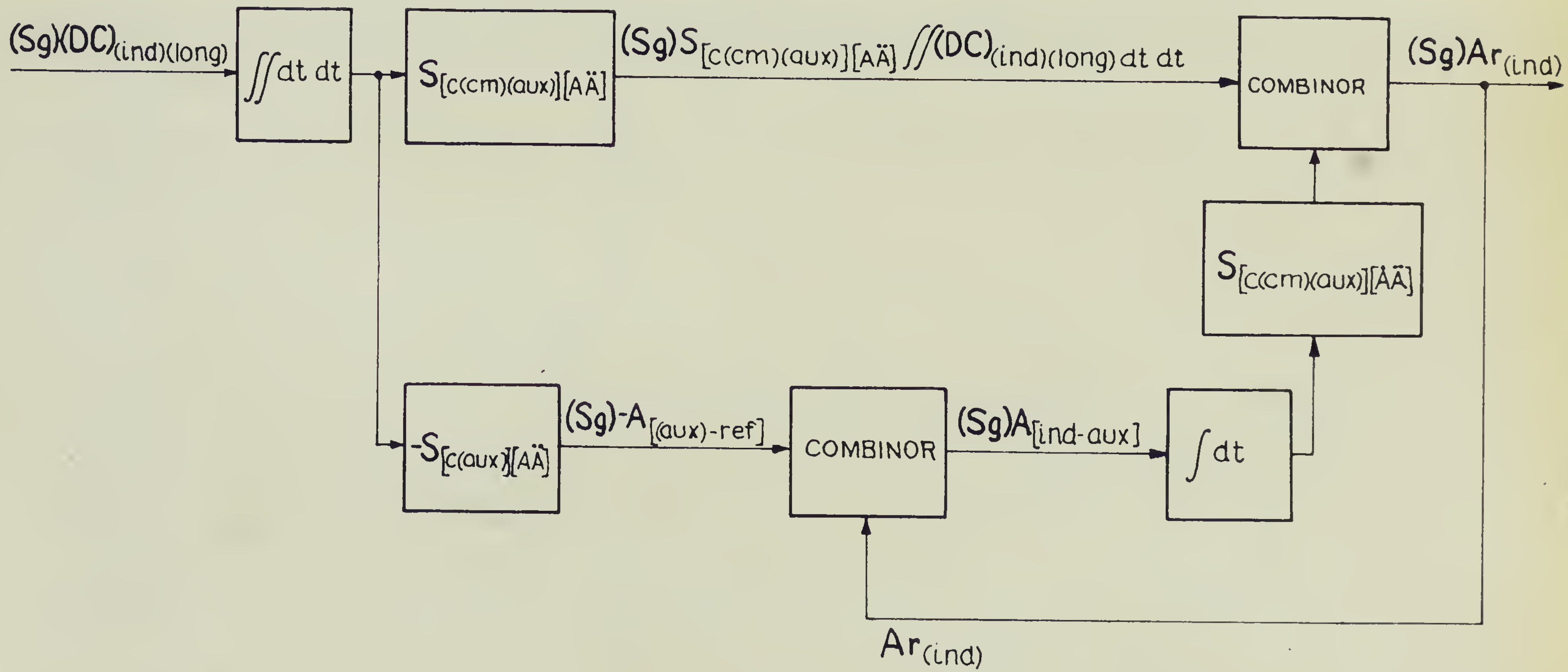


FIG. IV-2 BLOCK DIAGRAM IMPLEMENTATION OF CUBIC LONGITUDINAL MECHANIZATION EQUATION

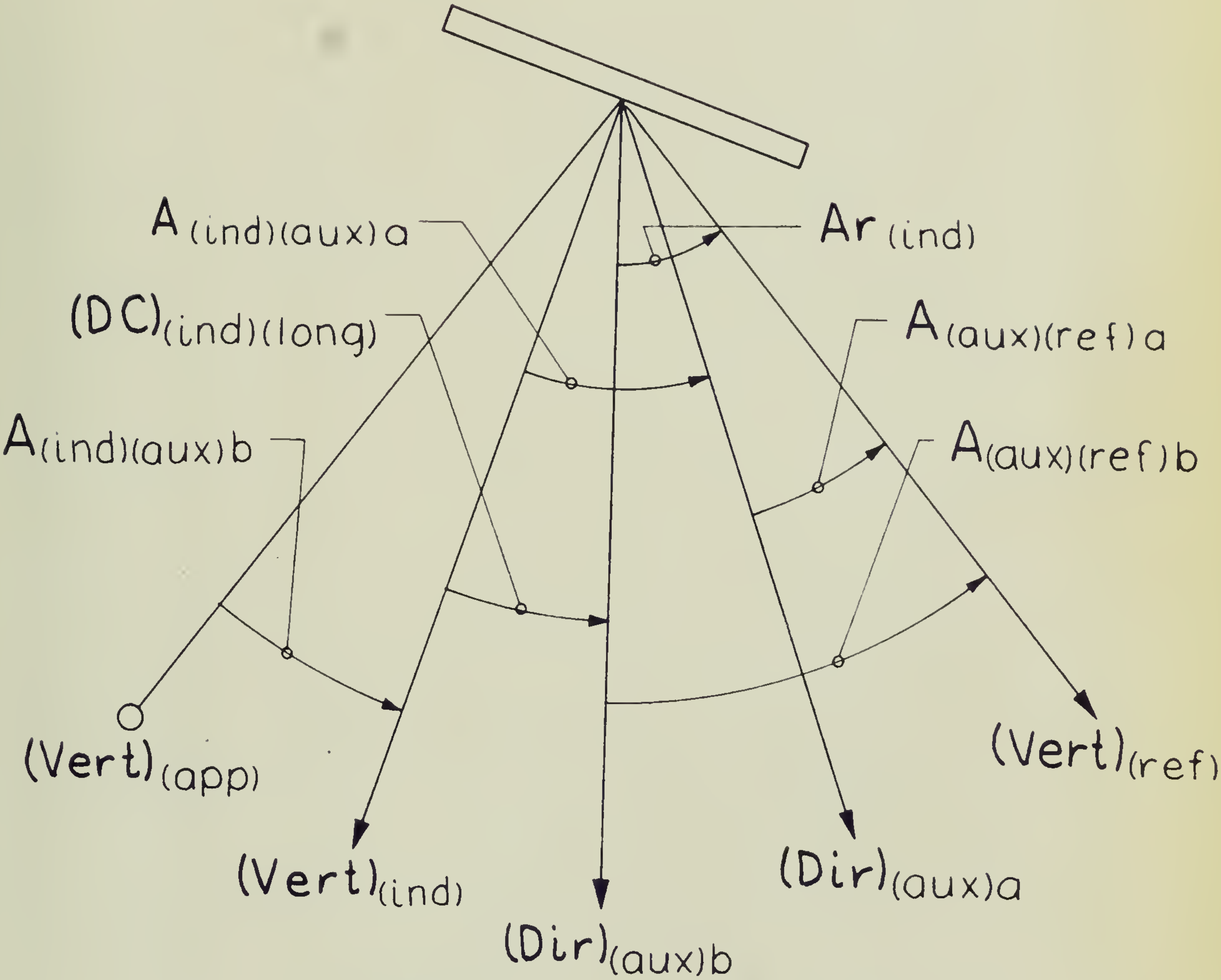


FIG. IV-3. AUXILIARY DIRECTIONS FOR QUARTIC OR QUINTIC MECHANIZATION EQUATION.

$$A_{(aux)(ref)b} = S_{[c(aux)b][\dot{A}\ddot{A}]} \iint (DC)_{(ind)long}\, dt\, dt \tag{IV-54}$$

The controlled member is positioned from the reference vertical through the use of the auxiliary directions by the equation

$$A_{r(ind)} = -S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int A_{(ind)(aux)a}\, dt - S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iint A_{(ind)(aux)b}\, dt\, dt \tag{IV-55}$$

These equations, when combined with the relations

$$A_{r(ind)} = A_{(aux)(ref)a} + A_{(ind)(aux)a} \tag{IV-56}$$

$$A_{r(ind)} = A_{(aux)(ref)b} + A_{(ind)(aux)b} \tag{IV-57}$$

become

$$A_{r(ind)} + S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int A_{r(ind)}\, dt + S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iint A_{r(ind)}\, dt\, dt = S_{[c(aux)a][\dot{A}\ddot{A}]}\, S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iiint (DC)_{(ind)long}\, dt\, dt\, dt + S_{[c(aux)b][\dot{A}\ddot{A}]}\, S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iiiint (DC)_{(ind)long}\, dt\, dt\, dt\, dt \tag{IV-58}$$

This is identical with the desired eq (IV-52) with the relationships among the sensitivities that

$$S_{[c(aux)a][\dot{A}\ddot{A}]} = \frac{S_{[c(cm)][\dot{A}A]}}{S_{[c(cm)][\ddot{A}A]}} \tag{IV-59}$$

$$S_{[c(aux)b][A\ddot{A}]} = \frac{1}{S_{[c(cm)][\ddot{A}A]}} \qquad \text{(IV-60)}$$

$$S_{[c(cm)(aux)][\dot{A}\ddot{A}]} = \frac{S_{[c(cm)][\dddot{A}A]}}{S_{[c(cm)][\ddddot{A}A]}} \qquad \text{(IV-61)}$$

$$S_{[c(cm)(aux)][A\dddot{A}]} = \frac{S_{[c(cm)][\dddot{A}A]}}{S_{[c(cm)][\ddddot{A}A]}} \qquad \text{(IV-62)}$$

Equations (IV-53), (IV-54), and (IV-55) can be mechanized almost as easily as were eqs (IV-45) and (IV-46), as shown in Fig IV-4. The choice between the cubic and quartic mechanization equations is largely, therefore, to be determined by a comparison of the dynamic responses of the system when these equations are used.

D. The method of mechanizing the quintic equation, (IV-36), is similar to that for the cubic and quartic equations just discussed. In integrated form, using $(DC)_{(ind)(long)}$ instead of $A_{r(app)}$, this equation becomes,

$$S_{[c(cm)][\ddddot{A}A]} A_{r(ind)} + S_{[c(cm)][\dddot{A}A]} \int A_{r(ind)}\, dt + S_{[c(cm)][\ddot{A}A]} \iint A_{r(ind)}\, dt\, dt + S_{[c(cm)][\dot{A}A]} \iiint A_{r(ind)}\, dt\, dt\, dt = S_{[c(cm)][\dot{A}A]} \iiiint (DC)_{(ind)long}\, dt\, dt\, dt\, dt + \int\!\!\int\!\!\int\!\!\int\!\!\int (DC)_{(ind)long}\, dt\, dt\, dt\, dt\, dt \qquad \text{(IV-63)}$$

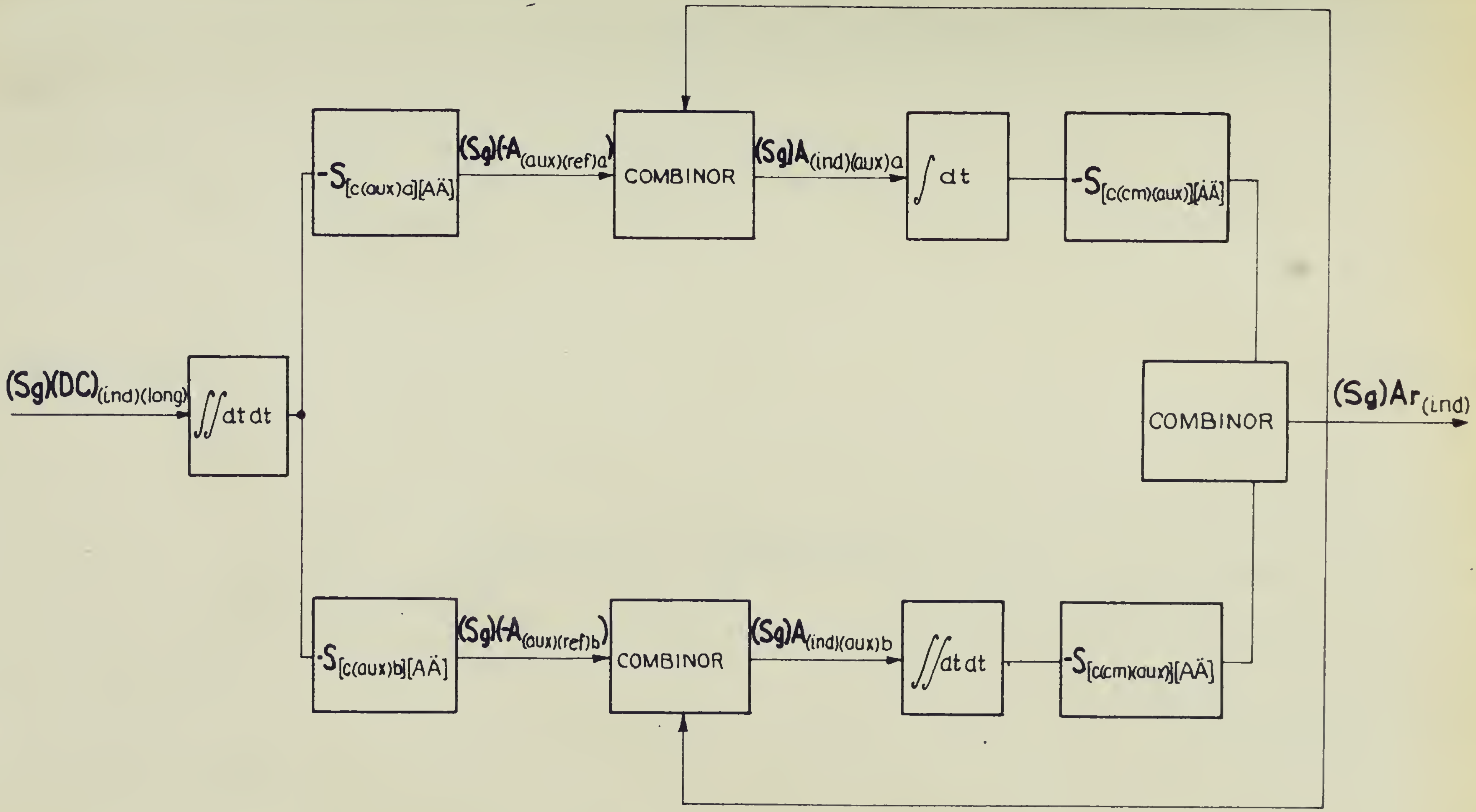


FIG. IV-4. BLOCK DIAGRAM OF IMPLEMENTATION OF QUARTIC LONGITUDINAL MECHANIZATION EQUATION.

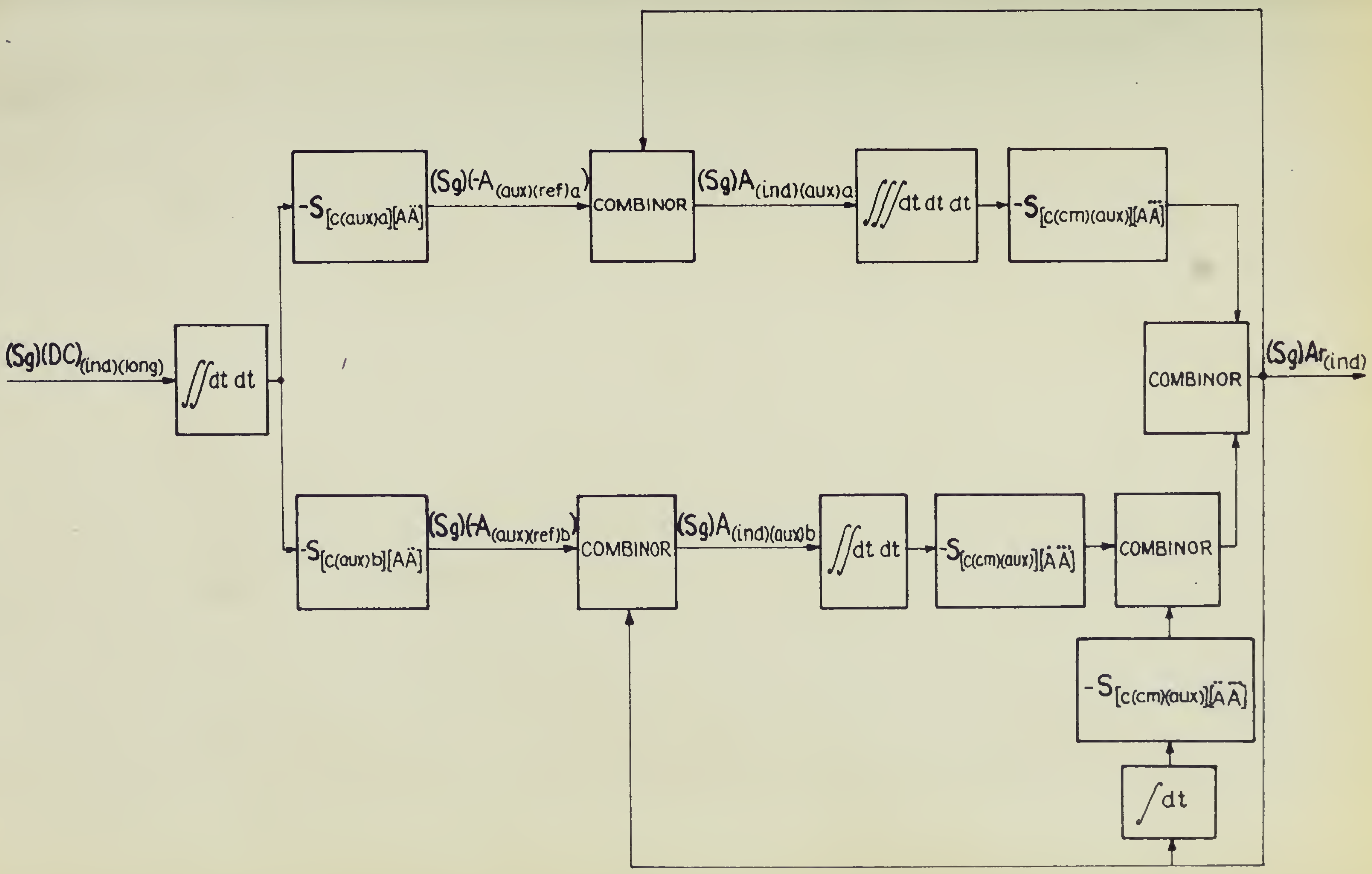


FIG. IV-5. BLOCK DIAGRAM OF IMPLEMENTATION OF QUINTIC LONGITUDINAL MECHANIZATION EQUATION.

The two auxiliary directions established by eqs (IV-53, 54, 56, and 57) are required to implement this equation.

The controlled member is positioned by the use of these equations and the equation

$$A_{r(ind)} = -S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iiint A_{(ind)(aux)a}\,dt\,dt\,dt - S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iint A_{(ind)(aux)b}\,dt\,dt - S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int A_{r(ind)}\,dt \tag{IV-64}$$

When these are combined, the control equation becomes

$$A_{r(ind)} + S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int A_{r(ind)}\,dt + S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iint A_{r(ind)}\,dt\,dt + S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iiint A_{r(ind)}\,dt\,dt\,dt = S_{[c(c)(aux)b][\dot{A}\ddot{A}]}\; S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \iiiint (DC)_{(ind)long}\,dt\,dt\,dt\,dt + S_{[c(aux)a][\dot{A}\ddot{A}]}\; S_{[c(cm)(aux)][\dot{A}\ddot{A}]} \int\!\!\int\!\!\int\!\!\int\!\!\int (DC)_{(ind)long}\,dt\,dt\,dt\,dt\,dt \tag{IV-65}$$

This is identical with the desired eq (IV-63). The mechanization of the equations from which (IV-65) is derived is shown in Fig IV-5. This mechanization is, of course, more complex than the systems previously discussed. Although the mechanization is entirely feasible, the quintic equation will not be considered further unless the dynamic responses of the system using the cubic and quartic control equations show themselves to be unsatisfactory.

4. The Closed-loop System

In order to examine the two mechanization equations between which choice is still to be made, it is necessary to examine the response of the closed-loop longitudinal system to wind forcing functions, using each of these mechanization equations. Response equations for particular mechanizations have been derived. The

general closed-loop equations are derived in this section.

The general mechanization equation can be written, for this study, as

$$A_{r(ind)} = (PF)_{(mech)}(DC)_{(ind)\,long} \tag{IV-66}$$

The linear distance of the missile measured along the track, from the initial reference point, can be written as

$$X_r = \iint \dot{V}_{[E-(air)]}\,dt\,dt + V_{[E-M](initial)}t + X_{(initial)} \tag{IV-67}$$

Also,

$$\frac{X_r}{R_E} = A_{r(true)} \tag{IV-68}$$

These relationships, with eqs (IV-1, 2 and 3) and the small-angle pendulum equation

$$(DC)_{(true)long} = \frac{\ddot{A}_{r(true)}R_E}{g_{IR}} = \frac{\dot{V}_{[E-(air)]}}{g_{IR}} \tag{IV-69}$$

combine to form the closed-loop diagram of Fig IV-6.

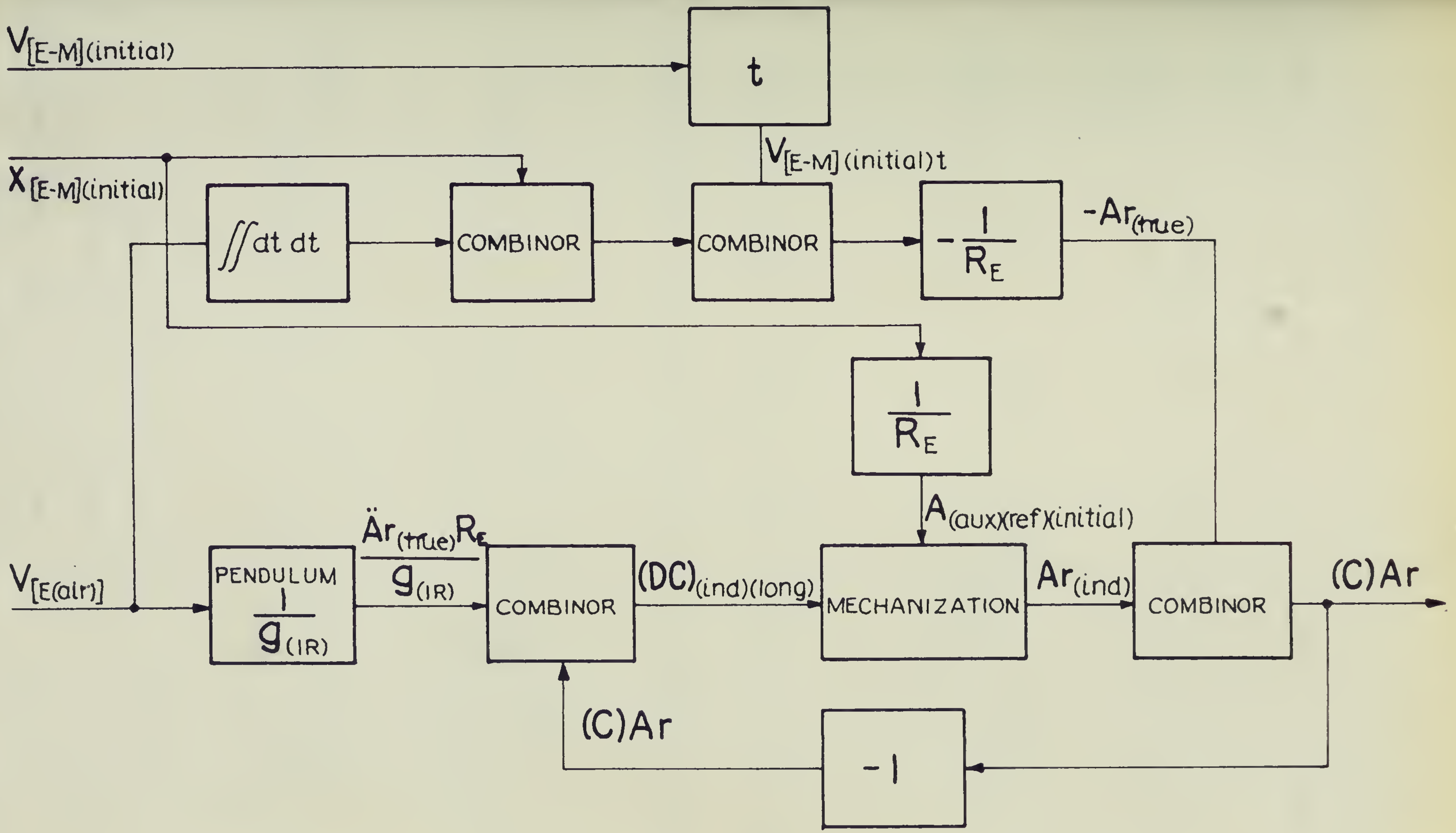


FIG. IV-6. RANGE INDICATION SYSTEM CLOSED-LOOP DIAGRAM.

The initial conditions in eq (IV-67) do not affect the response of $(C)A_r$ to $\dot{V}_{[E-(air)]}$, and can be taken as zero. Figure IV-6 can then be simplified to Fig IV-7.

In accordance with servomechanism theory, the overall performance equation of the range indication system can be written

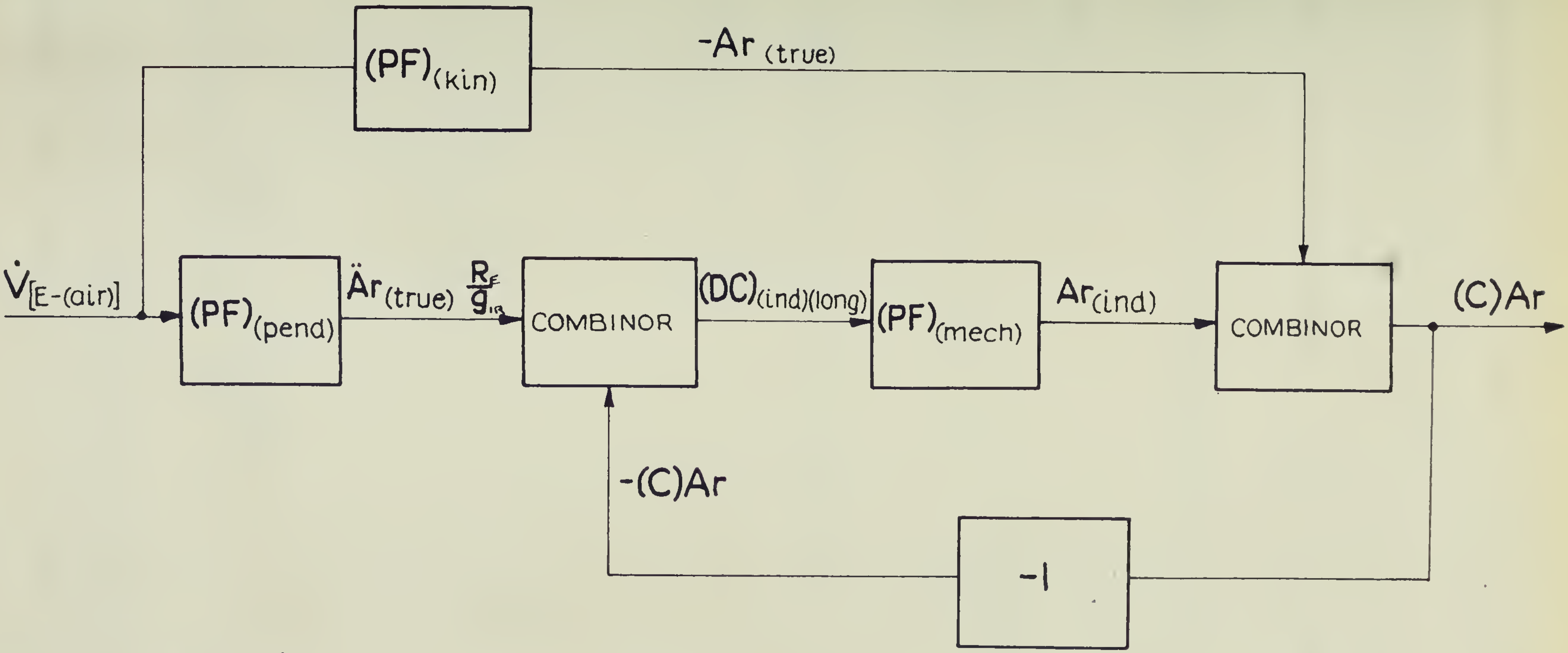
$$(PF)_{(ris)} = \frac{(C)A_r}{\dot{V}_{[E-(air)]}} = \frac{(PF)_{(pend)}(PF)_{(mech)} - (PF)_{(kin)}}{1 + (PF)_{(mech)}} \qquad \text{(IV-70)}$$

where $(PF)_{(pend)} = \frac{1}{g_{IR}}$ and $(PF)_{(kin)} = -\frac{1}{p^2 R_E}$. The mechanization equations to be examined can be written

$$(PF)_{(mech)(cubic)} = \frac{1 + S_{[c(cm)][\dot{A}A]}\,p}{S_{[c(cm)][\ddot{A}A]}\,p^2 + S_{[c(cm)][\dddot{A}A]}\,p^3} \qquad \text{(IV-71)}$$

$$(PF)_{(mech)(quartic)} = \frac{1 + S_{[c(cm)][\dot{A}A]}\,p}{S_{[c(cm)][\ddot{A}A]}\,p^2 + S_{[c(cm)][\dddot{A}A]}\,p^3 + S_{[c(cm)][\ddddot{A}A]}\,p^4} \qquad \text{(IV-72)}$$

Using these equations, numerical values can be assigned to the sensitivities, eq (IV-69)solved for various types of wind forcing functions, and the responses using eqs (IV-72) and (IV-71) compared. Numerical values are first assigned to the



$$(PF)_{(kin)} = -\frac{1}{P^2 R_E}$$

$$(PF)_{(pend)} = \frac{1}{g_{(IR)}}$$

FIG. IV-7 SIMPLIFIED RANGE CLOSED LOOP DIAGRAM.

cubic mechanization eq (IV-71), and then to the quartic eq (IV-72).

5. Selection of Numerical Values

A. The homogeneous control equation for the cubic system can be factored into a first order and a quadratic term, becoming

$$\left(p+\frac{1}{CT}\right)\left(p^2+2(DR)Wp+W^2\right)=0, \qquad \text{(IV-73)}$$

which expands into

$$p^3+\left[2(DR)W+\frac{1}{(CT)}\right]p^2+\left[W^2+\frac{2(DR)W}{(CT)}\right]p+\frac{W^2}{(CT)}=0 \qquad \text{(IV-74)}$$

Comparing coefficients term by term, with

$$S_{[c(cm)][\ddot{A}A]}=\frac{(CT)}{W^2} \qquad \text{(IV-75)}$$

$$(DR)=\frac{W^2-W_{NE}^2}{2W(CT)W_{NE}^2} \qquad \text{(IV-76)}$$

$$S_{[c(cm)][\acute{A}A]}=\left[(CT)+\frac{2(DR)}{W}\right] \qquad \text{(IV-77)}$$

Then, for any desired characteristic time and quadratic natural frequency, a damping ratio and set of sensitivities can be computed. Notice

that, once the characteristic time and the natural frequency have been selected, no control remains over the damping.

B. The homogeneous control equation for the quartic system can be factored into two quadratic terms. As in the track control system, these are made equal, giving,

$$(p^2+2(DR)Wp+W^2)(p^2+2(DR)Wp+W^2)=0 \qquad \text{(IV-78)}$$

This expands into

$$p^4+4(DR)Wp^3+2W^2[2(DR)^2+1]p^2+4(DR)W^3p+W^4=0 \qquad \text{(IV-78)}$$

Equating coefficients with those of the quartic control equation, with $S_{[C\ (cm)][\ddddot{A}\ A]} = \frac{1}{W^2_{NE}}$ in order to remove the acceleration term in the forcing function,

$$S_{[c(cm)][\dddot{A}\ A]} = \frac{1}{W^4} \qquad \text{(IV-80)}$$

$$(DR) = \sqrt{\frac{W^2-2W^2_{NE}}{4W^2_{NE}}} \qquad \text{(IV-81)}$$

$$S_{[c(cm)][\dot{A}\ A]} = \frac{4(DR)}{W} \qquad \text{(IV-82)}$$

$$S_{[c(cm)][\ddot{A}A]} = \frac{4(DR)}{W^3} \qquad \text{(IV-83)}$$

Plots follow for responses of the cubic and quartic control equations, with various sensitivities.

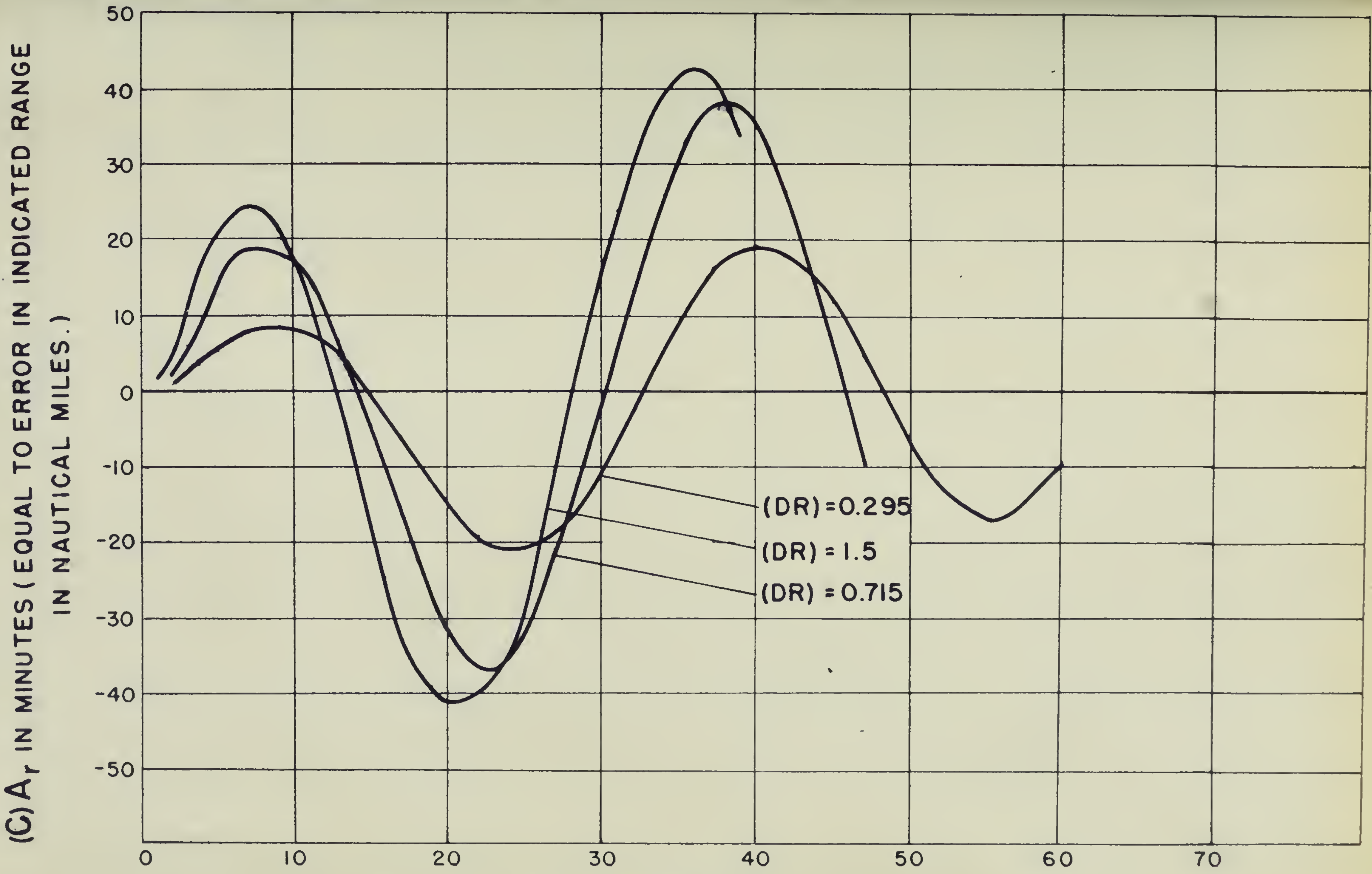


FIG. IV-8. RESPONSE OF CUBIC RANGE INDICATION SYSTEM TO 94.2 MPH. ROTATING WIND OF 30 MINUTE PERIOD.

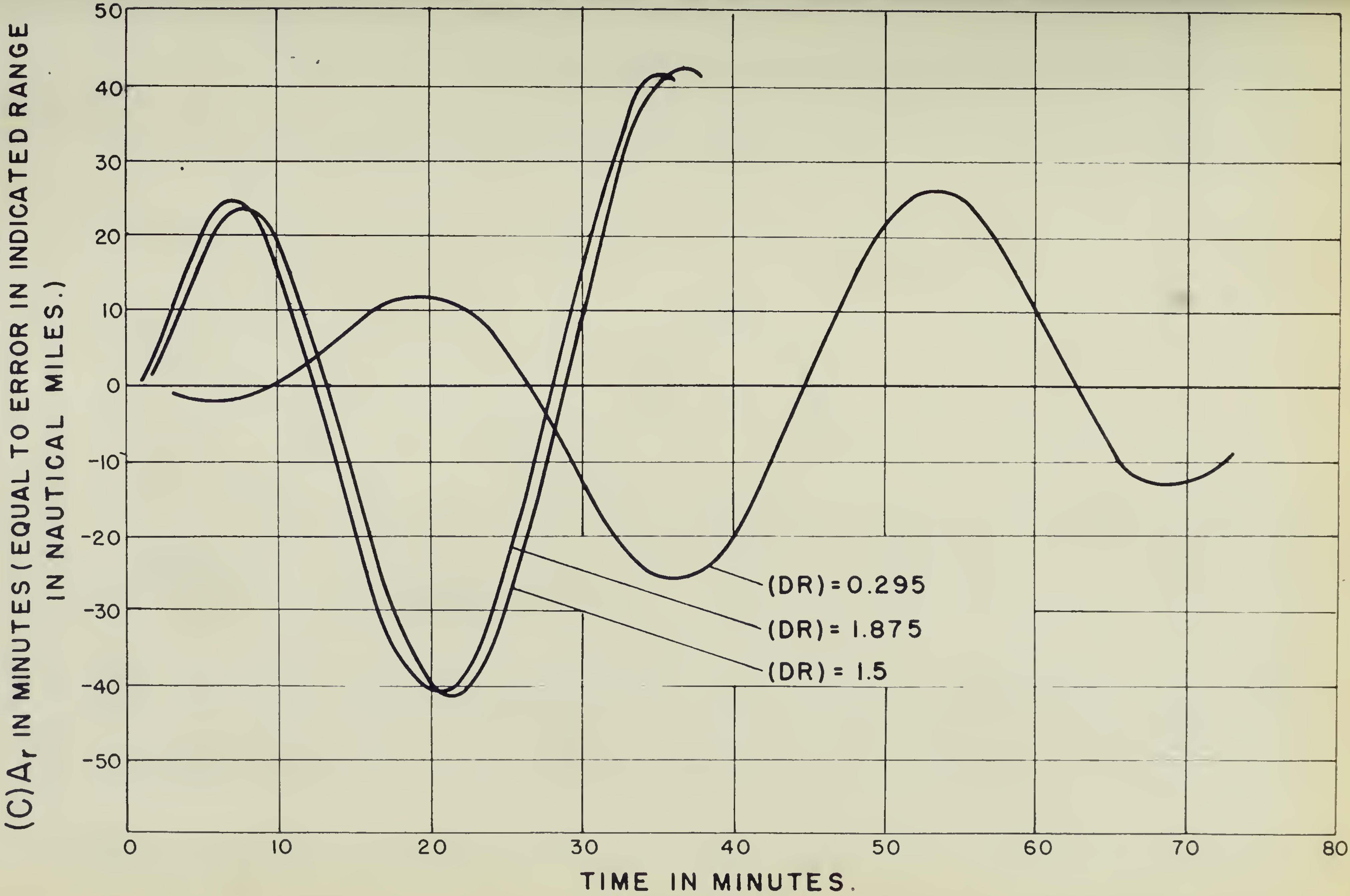


FIG. IV-9. RESPONSE OF QUARTIC RANGE INDICATION SYSTEM TO 94.2 MPH. ROTATING WIND OF 30MINUTE PERIOD.

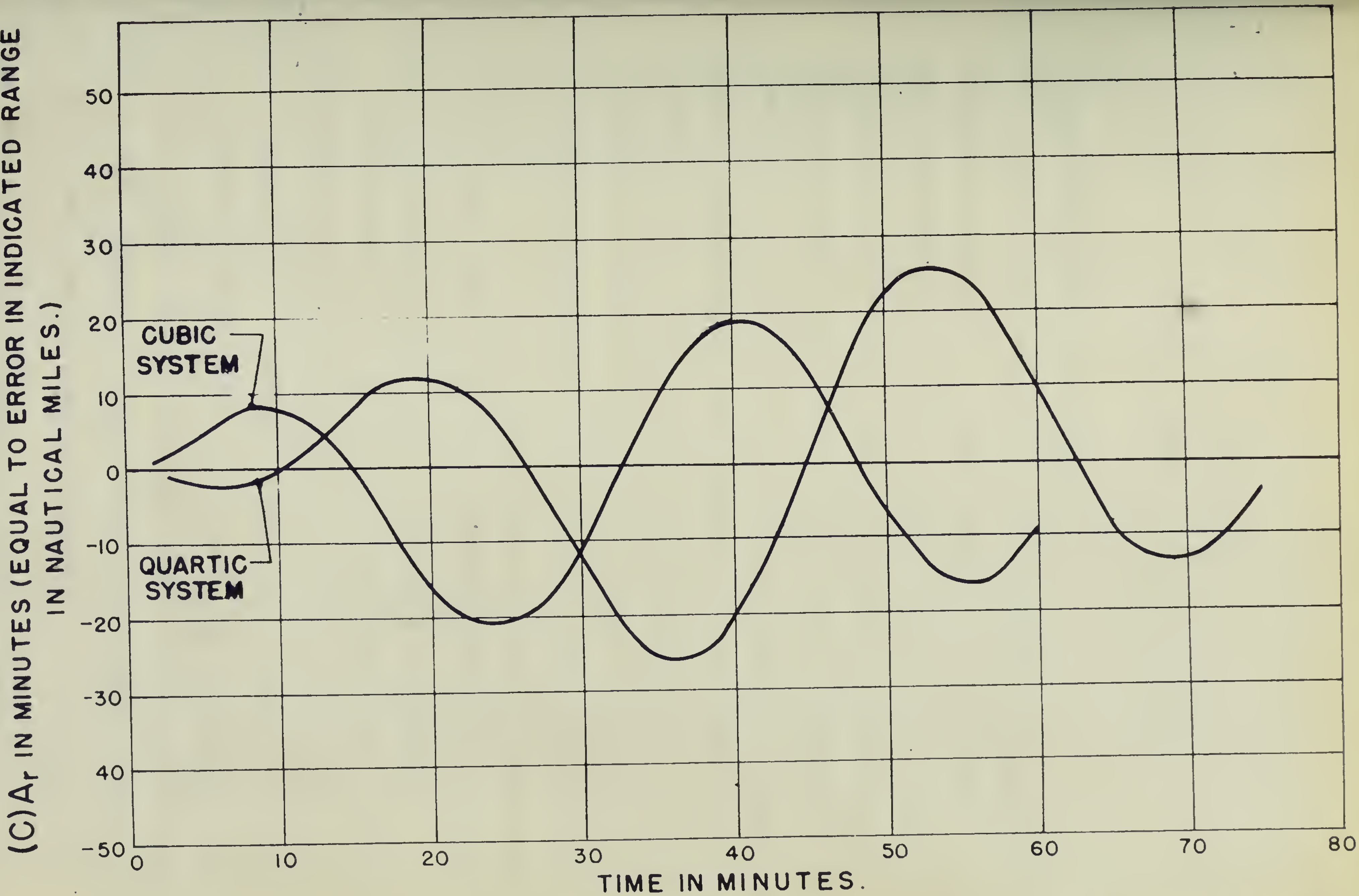


FIG. IV-10. COMPARISON OF CUBIC AND QUARTIC RANGE INDICATION SYSTEM RESPONSE (DR)=0.295.

CHAPTER V

CONCLUSIONS AND SUGGESTIONS

1. Disturbances to the Control Systems

In examining the success of the mechanization systems whose responses have been analyzed in the preceding chapters, it is necessary to consider the types of disturbing inputs which will affect the systems. These disturbances, or winds, are of three major types. First, there are gust disturbances of fairly high frequency and short duration. These are enormously attenuated by the long time constants of the systems analyzed, and should cause no trouble. Second, there are acceleration pulse type inputs, such as would be encountered when crossing weather fronts, where the wind velocity changes very rapidly to a finite and fairly constant value. These can be analyzed by the step function responses of Chapters III and IV. In Fig V-1, the path through the "eye" of a hurricane has been approximated with step functions of acceleration, and in Fig V-2, the response of the track control system is shown to be satisfactory, even with this violent disturbance.

There is, however, a third type of disturbance; that in which a wind of approximately constant velocity slowly rotates.* This condition has been approximated by sinusoidal inputs as shown in Chapters III and IV. The error if the sinusoidal forcing function continues for a long period of time, is seen to be very large. A wind of about forty knots, rotating with a period of about one hour, will introduce, in the steady state, deviations from the track, in the track control system, of about 20 miles.

*Ref. A. G. Bogosian Notebook dated March 18, 1947.

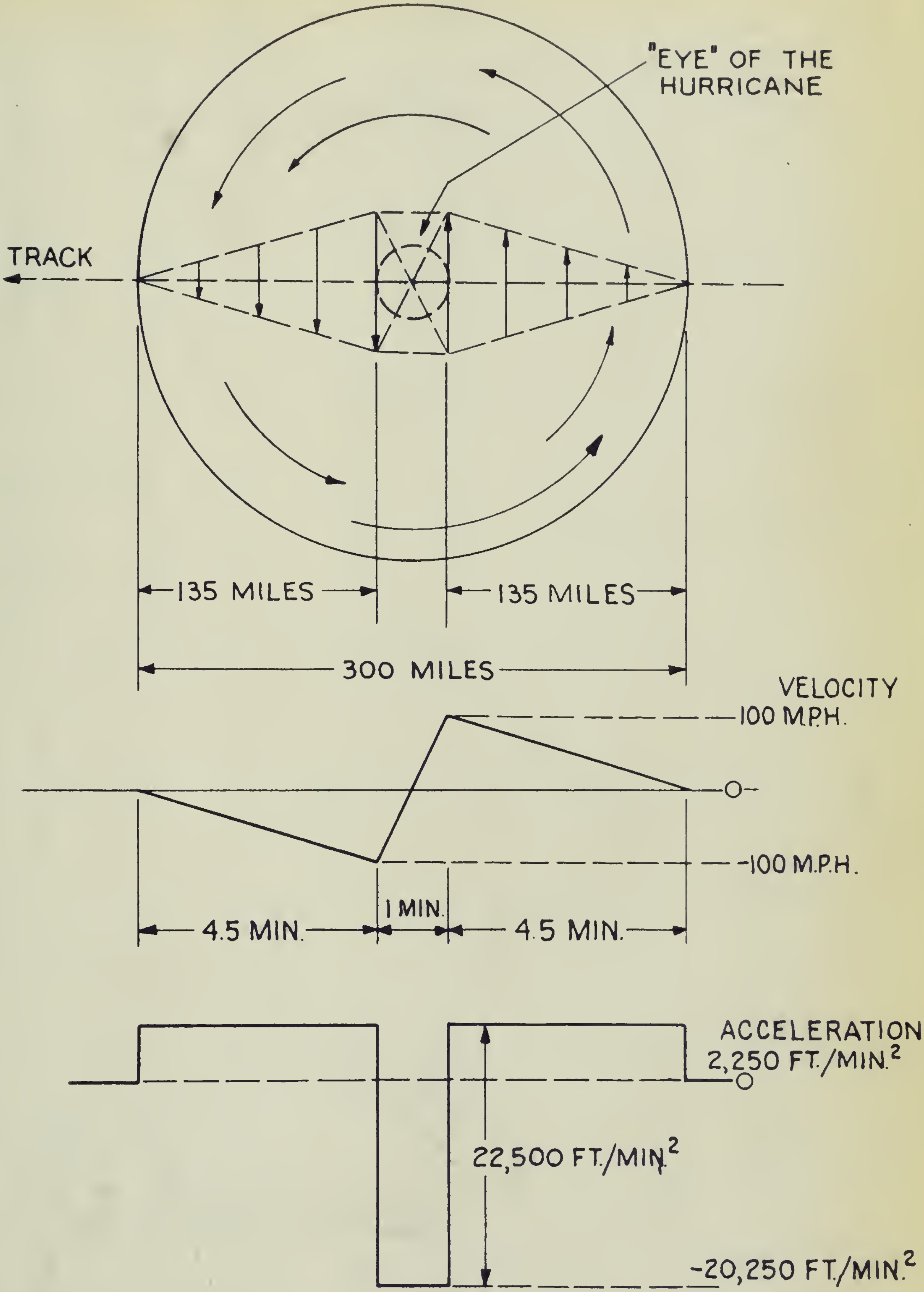


FIG. V-1. APPROXIMATION OF A HURRICANE USING ACCELERATION PULSES.

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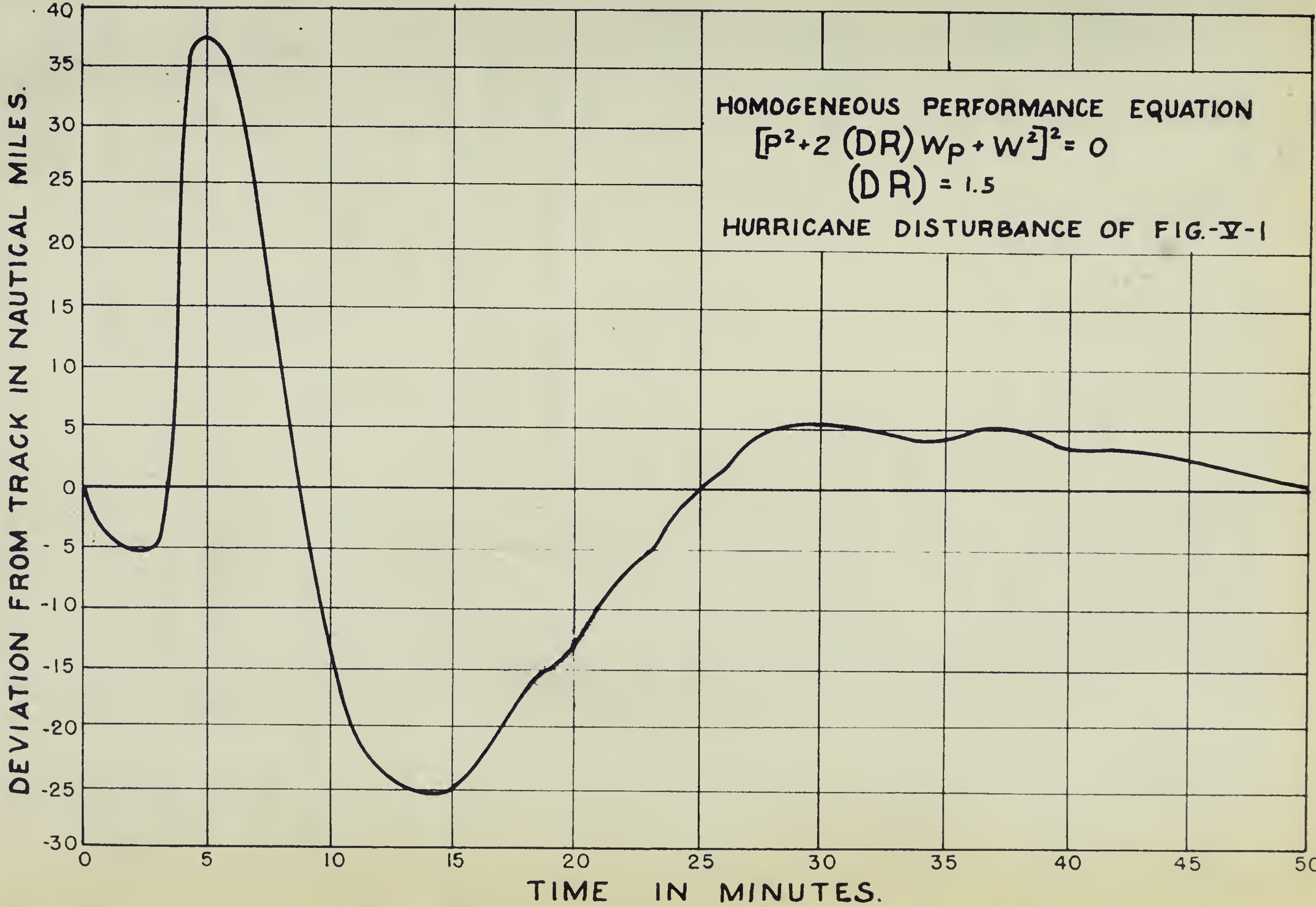

FIG.-V-2 RESPONSE OF TRACK CONTROL SYSTEM TO IDEALIZED HURRICANE.

Three ways of correcting this condition suggest themselves:

a. Build systems with variable coefficients, which provide the necessary high damping with step-inputs, and have a different response to such low-acceleration inputs as sinusoidally varying winds.

b. Increase the order of the track control system so that the forced error results only from high order greatly attenuated terms.

c. Modify the system to reduce the effects of the forced errors.

These three methods are all examined in the next section of this chapter.

2. <u>Suggestions for Improvement of System Response</u>

A. Track Control

a. A variable coefficient system can be of value only if changing the damping and period of the system will improve the response. Since the coefficient of the acceleration rate forced error term increases rapidly with increase in damping, it appears that this should be possible. A series of track control responses using the same magnitude of disturbance but different damping ratios has therefore been examined, Fig. V-3. It is seen that no choice of damping is satisfactory. If damping is entirely removed, the forced error of the simplified system reduces to zero. Actually, a forced error will still exist if there is any vertical wind acceleration, or if non-ideal components are used. Then, if the frequency of the rotating wind is equal to that of the system, very large errors can result. This approach, therefore, does not provide the needed improvement of response.

b. The purpose of this section is to investigate the response of a quintic performance equation. From eq III-26 the performance equation may be written:

(DR)	PERIOD OF ROTATING WIND	(C) $A_{(Vert)(tc)(max)}$ IN MINUTES OF ARC [EQUAL TO NAUTICAL MILES OFF TRACK]
1.875	10	7.735
1.875	15	14.310
1.500	15	13.620
1.050	15	7.600
1.875	20	13.033
1.875	30	9.548
1.500	30	14.899
1.050	30	13.652
1.500	40	12.080
1.050	40	14.096

MAXIMUM TRACK ERROR FOR VARIOUS (DR)s WITH ROTATING WINDS OF 31.4 MPH

FIGURE V-3

$$\frac{(C)A\,(Vert)_{(tc)}}{\dot{V}_{[E-(air)]}} = \frac{\left\{\dfrac{W_{NE}^{2}}{S_{[C(CD)][\dot{A}\ddot{A}]}\,V_{[(air)-M]}}\right\}p^{2}}{p^{5}+\left\{\dfrac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}\,V_{[(air)-M]}}+\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right\}p^{4}+\left\{W_{NE}^{2}+\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right\}p^{3}+\left\{W_{NE}^{2}\,\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}+\dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right\}p^{2}+\left\{W_{NE}^{2}\,\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right\}p+\left\{W_{NE}^{2}\,\dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right\}}$$

For this study it is assumed that the denominator of the performance equation may be factored into the form:

$$\left(p + \frac{1}{CT}\right)\left(p^2 + 2(DR)W\,p + W^2\right)^2$$

Let $\quad W = K\,W_{NE} \qquad$ where $\quad W_{NE} \triangleq \sqrt{\dfrac{g_{IR}}{R_E}}$

Then expanding and equating coefficients of like items,

$$4K(DR)W_{NE} + \frac{1}{CT} = \frac{g_{IR}}{S_{[C(CD)][\ddot{A}\,\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[C(CD)][\ddot{A}\,\dddot{A}]}}{S_{[C(CD)][\dddot{A}\,\dddot{A}]}} \qquad \text{V-2}$$

$$2K^2\,W_{NE}^2 + 4K^2(DR)^2\,W_{NE}^2 + \frac{4K(DR)W_{NE}}{CT} = W_{NE}^2 + \frac{S_{[C(CD)][\dot{A}\,\dddot{A}]}}{S_{[C(CD)][\ddot{A}\,\ddot{A}]}} \qquad \text{V-3}$$

$$\frac{2K^2\,W_{NE}^2 + 4K^2(DR)^2W_{NE}^2}{CT} + 4K^3(DR)W_{NE}^3 = W_{NE}^2\,\frac{S_{[(C)(CD)][\ddot{A}\,\dddot{A}]}}{S_{[(C)(CD)][\ddot{A}\,\dddot{A}]}} + \frac{S_{[C(CD)][A\,\dddot{A}]}}{S_{[C(CD)][\ddot{A}\,\ddot{A}]}} \qquad \text{V-4}$$

$$\frac{4K^3(DR)W_{NE}^3}{CT} + K^4\,W_{NE}^4 = W_{NE}^{\varepsilon}\,\frac{S_{[C(CD)][\dot{A}\,\dddot{A}]}}{S_{[C(CD)][\dddot{A}\,\ddot{A}]}} \qquad \text{V-5}$$

$$\frac{K^4 W_{NE}^4}{CT} = W_{NE}^2 \frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\dddot{A}]}} \qquad \text{(V-6)}$$

Examination of eqs (V-2, 3, 4, 5, and 6) reveals that if K and (DR) are selected, then CT and all the sensitivities are fixed. General expressions are now developed for determining the numerical values of CT and of certain ratios of sensitivities.

From simultaneous solution of eqs (V-3 and 5)

$$CT = \frac{4K(DR)(K^2-1)}{W_{NE}(-K^4+6K^2-1)} \qquad \text{(V-7)}$$

From eq (V-6)

$$\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\dddot{A}]}} = \frac{K^3 W_{NE}^3(-K^4+6K^2-1)}{4(DR)(K^2-1)} \qquad \text{(V-8)}$$

From eqs (V-5 and 7),

$$\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\dddot{A}]}} = \frac{K^2 W_{NE}^2(5K^2-1)}{K^2-1} \qquad \text{(V-9)}$$

From eqs (V-4, 6 and 7),

$$\frac{S_{[C(CD)][\ddot{A}\dddot{A}]}}{S_{[C(CD)][\dot{A}\ddot{A}]}} =$$

$$W_{NE}\left\{\frac{K^7+4K^5[3(DR)^2-2]+K^3[13+8(DR)^2]-2K[1+2(DR)^2]}{4(DR)(K^2-1)}\right\} \qquad \text{(V-10)}$$

From eqs (V-2, 4 and 6)

$$\frac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} = \left\{\frac{-K^8+4K^6[2-3(DR)^2]+2K^4[-7+4(DR)^2]-4K^2[2-3(DR)^2]-1}{4K(DR)(K^2-1)}\right\} \quad \text{V-11}$$

Limited time permits the evaluation of only one numerical solution of the performance equation.

Let K = 2 and (DR) = 1.

Then,

$$\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} = \frac{14\,W_{NE}^3}{3}$$

$$\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\dot{A}\ddot{A}]}} = \frac{76\,W_{NE}^2}{3}$$

$$\frac{S_{[C(CD)][\ddot{A}\dot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} = \frac{103\,W_{NE}}{3}$$

$$\frac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}} = -\frac{625}{24}\,W_{NE}$$

$$\frac{1}{CT} = \frac{7}{24}\,W_{NE}$$

Substituting these values into the performance equation and using as an input $\dot{V}_{[E\,-\,(air)]}$ = 2000 ft/min^2 $\sin \frac{2\pi}{15}\,t$,

(C) A (Vert) (tc) = .8 minutes of arc (.8 nautical miles)

Comparing this answer to those obtained in the plot of results in Chapter III indicates that the quintic performance equation produces a better response than the quartic one. It should be mentioned that no effort has been made to select optimum parameters for the quintic.

c. It is seen, by an examination of the track control system performance function, that no control over the gain setting exists. As was shown in Chapter III, it is desirable to have the sensitivities negative for the quartic performance equation. Making this change, eq III-25 becomes;

$$(PF)_{(tc)(CL)[\dot{V}A]} = \frac{p \dfrac{W_{NE}^2}{S_{[C(CD)][\ddot{A}\ddot{A}]} V_{[(air)-M]}}}{p^4 + \left[\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} + \dfrac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]} V_{[(air)-M]}}\right] p^3 + \left[\dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} + W_{NE}^2\right] p^2 + \dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} W_{NE}^2\, p + \dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} W_{NE}^2} \qquad \text{V-12}$$

where sensitivities are now positive numbers.

If a controllable gain coefficient could be introduced into the numerator, so that an attenuation of the forced error could be provided, then the system might be made to perform without excessive errors. This could be done, it appears, if the wind acceleration term could be attenuated after entering the system. This cannot be done, but an approximation of this desired result is achieved if additional feedback loops are provided as indicated in figure V-4. The performance equation for this system, when $(PF)_{(aero)(ind)}$ is assumed to be exactly equal to $(PF)_{(aero)[AA]}$

$$(PF)_{(tc)(CL)[\dot{V}A]} = \frac{\left\{\left[(PF)_{(aux)[AA]} - 1\right](PF)_{(pu)[\dot{V}A]}(PF)_{(aero)[AA]}(PF)_{(mech)[AA]} + (PF)_{(pu)[\dot{V}A]}\right\}\left\{(PF)_{G[AA]} - 1\right\}}{1 + (PF)_{(aero)[AA]}(PF)_{(mech)[AA]}\left[(PF)_{(aux)[AA]} - 1 - (PF)_{(aux)[AA]}(PF)_{G[AA]}\right]} \qquad \text{V-13}$$

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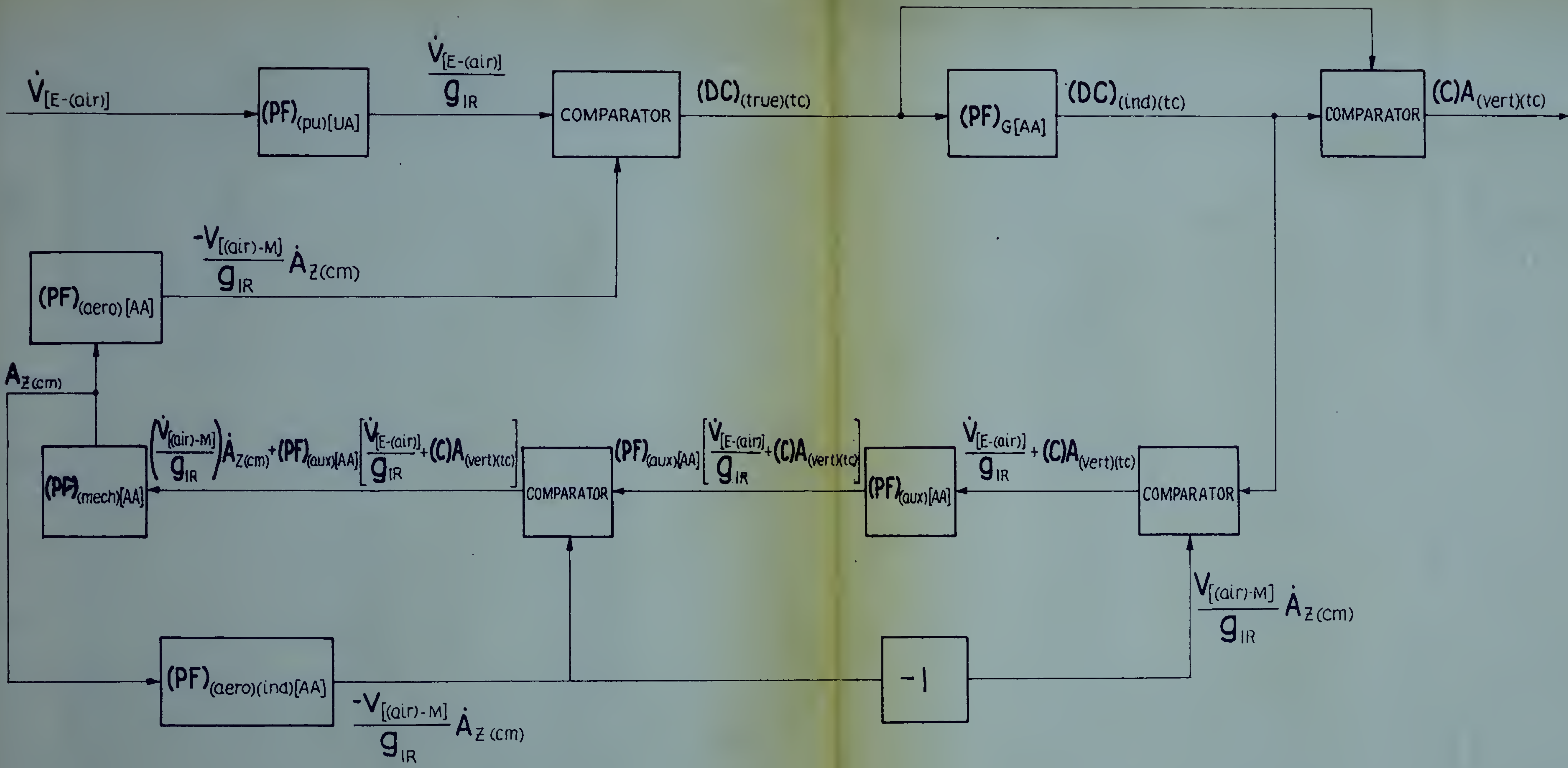

$$(PF)_{(pu)[VA]} = \frac{1}{g_{IR}} \qquad (PF)_{(aero)[AA]} = \frac{-V_{[(air)-M]}}{g_{IR}}\,p \qquad (PF)_{(aux)[AA]} = S_{(aux)[AA]} + S_{(aux)[A\dot{A}]}\,p$$

$$(PF)_{G[AA]} = \frac{p^2+W^2_{NE}}{p^2} \qquad (PF)_{(aero)(ind)[AA]} = \frac{-V_{[(air)-M](ind)}}{g_{IR}}\,p \qquad (PF)_{(mech)} = \text{MECHANIZATION FROM CHAPTER THREE}$$

FIG. V-4. MODIFIED TRACK CONTROL SYSTEM.

when the derivative auxiliary feedback sensitivity is set at zero, this becomes, with proportional plus first and second integral mechanization equation,

$$(PF)_{(tc)(CL)[\dot{V}} = \frac{\dfrac{-S_{(aux)[AA]}-1}{R_E}\,p^2 - \left[\dfrac{S_{[C(CD)][\dot{A}\dot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\left(\dfrac{S_{(aux)[AA]}-1}{R_E}\right) + \dfrac{1}{R_E}\left(\dfrac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right)\right]p - \dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\left(\dfrac{S_{(aux)[AA]}-1}{R_E}\right)}{p^4 + \left(\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} - \dfrac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right)p^3 + \left(\dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} + S_{(aux)[AA]}W^2_{NE}\right)p^2 + S_{(aux)[AA]}\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}W^2_{NE}\,p + S_{(aux)[AA]}\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}-1}{S_{[C(CD)][\ddot{A}\ddot{A}]}}W^2_{NE}} \qquad \text{V-14}$$

In this equation, a certain amount of gain control has been introduced into the numerator, at the expense of introducing a term giving a forced error caused by wind acceleration. Routh's stability criteria show that this system, with proper selection of sensitivities, is stable. Limitations of time did not permit the authors to make a complete analysis of this control system. However, a sample result, selected at random is calculated. The denominator is first factored into two unequal quadratics, and the coefficients squared. They become:

$$\left(\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) = \frac{W_1^2\,W_2^2}{S_{(aux)[AA]}W^2_{NE}} \qquad \text{V-15}$$

$$\left(\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) + S_{(aux)[AA]}W^2_{NE} = [W_1^2 + W_2^2 + 2(DR)_1(DR)_2\;\; W_1W_2 \qquad \text{V-16}$$

$$\left(\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) = \frac{2(DR)_1W_1W_2^2 + (DR)_2W_1^2W_2}{S_{(aux)[AA]}W^2_{NE}}$$

$$\left(\frac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right) = \left(\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) - 2[(DR)_1W_1 + (DR)_2W_2] \qquad \text{V-17}$$

The values chosen were

$S_{(aux)[AA]} = 0.1$

$W_1 = 1.5W_{NE}$

$W_2 = W_{NE}$

$(DR)_2 = 0.7$

These gave, using the equations given above, $(DR)_1 = 9.18$,

$$\left(\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) = 0.124,\ \left(\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) = 22.84 \text{ and } \left(\frac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right) = 20.72$$

The magnitude of the maximum forced error was found to be 28.5 nautical miles, and the disturbance which gave this maximum deviation was a sinusoidally varying wind with a velocity magnitude of 265.5 nautical miles per hour. This is a considerable improvement over the uncompensated system.

Next, $S_{(aux)[AA]}$ was set at unity to remove the accleration term, and $S_{(aux)[AA]}$ was set, at a random value.

The performance equation, with quartic mechanization, becomes

$$(PF)_{(lc)(CL)[\dot{V}A]} = \frac{-\frac{S_{(aux)[\dot{A}A]}}{R_E}p^3 - \frac{S_{(aux)[\dot{A}A]}}{R_E}\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}p^2 - \left(\frac{S_{[C(CD)][\dot{A}A]}}{R_E}\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} - \frac{g_{IR}}{R_E S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right)p}{p^4 + \left(\frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} + S_{(aux)[\dot{A}A]}W_{NE}^2 - \frac{g_{IR}}{V_{[(air)-M]}S_{[C(CD)][\ddot{A}\ddot{A}]}}\right)p^3 + \left(\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}} + \frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}S_{(aux)[\dot{A}A]}W_{NE}^2 + W_{NE}^2\right)p^2 + \left(\frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}S_{(aux)[\dot{A}A]}W_{NE}^2 + \frac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}W_{NE}^2\right)p + \frac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}W_{NE}^2} \qquad \text{V - 18}$$

Equating coefficients, as before, to those of a quartic with the desired characteristics, and selecting, at random, the following values:

$S_{(aux)[\dot{A}A]} = 20$

$W_1 = W_2 = 2W_{NE}$

Answers were obtained making $\left(\dfrac{S_{[C(CD)][A\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) = .0885$

$(DR)_1 = (DR)_2 = 2.75$, $\left(\dfrac{S_{[C(CD)][\dot{A}\ddot{A}]}}{S_{[C(CD)][\ddot{A}\ddot{A}]}}\right) = 4.78$ and $\left(\dfrac{g_{IR}}{S_{[C(CD)][\ddot{A}\ddot{A}]}V_{[(air)-M]}}\right) = 3.25$

With the same wind input as that used in the preceding example, the maximum error was found to be 48.8 miles. This, again, is an improvement over the uncompensated system.

It appears, since in the first example the error was caused chiefly by the acceleration term, and in the second example by the second time rate of change of acceleration, that a combination of the two systems, varying both $S_{(aux)[AA]}$ and $S_{(aux)[\dot{A}A]}$ might give a satisfactory solution. Time prevents such a study here. A simulator would greatly facilitate the work of selecting optimum sensitivities.

B. Range Indication

Lack of time prevented an elaborate examination of the response of the range indication system to large ranges of inputs. Since the track control system indicates that the critical response is to sinusoidal inputs, a series of solutions was made for cubic and quartic responses to a sinusoidal acceleration input with a period of thirty minutes.

As indicated in the plots of Chapter IV, the response is improved as the damping ratio of the quadratic in the cubic equation, or the two equal quadratics in the quartic equation, is decreased. At a damping ratio near 0.3,

3. TRACK CONTROL MECHANISM SUMMARY

MECHANIZATION EQUATION	PERFORMANCE EQUATION AND REMARKS
$A_{z(Cm)} = S_{[C(CD)][AA]}(DC)_{(ind)(tc)}$	$(C)\ddot{A}_{(Vert)(tc)} + \frac{R_E W_{NE}^2}{S_{[C(CD)][AA]} V_{[(air)\cdot M]}}(C)\dot{A}_{(Vert)(tc)} + W_{NE}^2 (C)A_{(Vert)(tc)} = \frac{W_{NE}^2}{S_{[C(CD)][AA]} V_{[(air)-M]}} V_{[E-(air)]}$ VELOCITY OF WIND INTRODUCES FORCED ERROR
$A_{z(Cm)} = -S_{[C(CD)][A\dot{A}]}\int(DC)_{(ind)(te)}\,dt$	$(C)\ddot{A}_{(Vert)(tc)} + \frac{W_{NE}^2}{\frac{g_{IR}}{S_{[C(CD)][A\dot{A}]}}+1}(C)A_{(Vert)(tc)} = \frac{W_{NE}^2}{g_{IR} + S_{[C(CD)][A\dot{A}]} V_{[(air)-M]}} \dot{V}_{[E-(air)]}$ THIS SYSTEM HAS NO DAMPING AND BECOMES UNSTABLE IF ATTEMPT IS MADE TO ELIMINATE FORCED ERROR INTRODUCED BY ACCELERATION OF THE WIND.
$A_{z(Cm)} = -S_{[C(CD)][A\dot{A}]}\iint(DC)_{(ind)(tc)}\,dt\,dt$	$(C)\ddot{A}_{(Vert)(tc)} + \frac{S_{[C(CD)][A\ddot{A}]} V_{[(air)-M]}}{g_{IR}}(C)\ddot{A}_{(Vert)(tc)} + \frac{S_{[C(CD)][A\ddot{A}]} V_{[(air)-M]}}{R_E}(C)A_{(Vert)(tc)} = \frac{1}{R_E}\ddot{V}_{[E-(air)]}$ THIS SYSTEM IS UNSTABLE BECAUSE THE FIRST ORDER TERM IS MISSING

MECHANIZATION EQUATION	PERFORMANCE EQUATION AND REMARKS
$\int A_{z(cm)}\,dt = S_{[c(CD)][\dot{A}A]}(DC)_{(ind)(tc)}$	$(C)\dddot{A}_{(Vert)(tc)} + W^2_{NE}\left\{1+\frac{R_E}{S_{[c(CD)][\dot{A}A]}V_{[(air)-M]}}\right\}(C)\dot{A}_{(Vert)(tc)} = \frac{W^2_{NE}}{S_{[c(CD)][\dot{A}A]}V_{[(air)-M]}}V_{[E-(air)]}$ THIS SYSTEM IS UNSTABLE BECAUSE CONSTANT TERM IS MISSING
$A_{z(cm)} = -S_{[c(CD)][\ddot{A}\dot{A}]}(DC)_{(ind)(tc)} - S_{[c(CD)][\dot{A}A]}\int(DC)_{(ind)(tc)}\,dt$	$(C)\dddot{A}_{(Vert)(tc)} + W^2_{NE}\left\{\frac{R_E}{V_{[(air)-M]}} + \frac{S_{[c(CD)][\dot{A}A]}}{W^2_{NE}}\right\}(C)\ddot{A}_{(Vert)(tc)} + W^2_{NE}(C)\dot{A}_{(Vert)(tc)} + W^2_{NE}\frac{S_{[c(CD)][\dot{A}A]}}{S_{[c(CD)][\ddot{A}\dot{A}]}}(C)A_{(Vert)(tc)} = \frac{W^2_{NE}}{S_{[c(CD)][\ddot{A}\dot{A}]}V_{[(air)-M]}}\dot{V}_{[E-(air)]}$ ACCELERATION OF THE WIND INTRODUCES A FORCED ERROR.
$A_{z(cm)} = -S_{[c(CD)][\dddot{A}\ddot{A}]}(DC)_{(ind)(tc)} - S_{[c(CD)][\ddot{A}\dot{A}]}\int(DC)_{(ind)(tc)}\,dt$ $- S_{[c(CD)][\dot{A}A]}\iint(DC)_{(ind)(tc)}\,dt\,dt$	$(C)\ddddot{A}_{(Vert)(tc)} + \left\{\frac{g_{1R}}{S_{[c(CD)][\dddot{A}\ddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\ddot{A}\dot{A}]}}{S_{[c(CD)][\dddot{A}\ddot{A}]}}\right\}(C)\dddot{A}_{(Vert)(tc)} + \left\{W^2_{NE} + \frac{S_{[c(CD)][\dot{A}A]}}{S_{[c(CD)][\dddot{A}\ddot{A}]}}\right\}(C)\ddot{A}_{(Vert)(tc)} + W^2_{NE}\frac{S_{[c(CD)][\ddot{A}\dot{A}]}}{S_{[c(CD)][\dddot{A}\ddot{A}]}}(C)\dot{A}_{(Vert)(tc)} + W^2_{NE}\frac{S_{[c(CD)][\dot{A}A]}}{S_{[c(CD)][\dddot{A}\ddot{A}]}}(C)A_{(Vert)(tc)} = \frac{W^2_{NE}}{S_{[c(CD)][\dddot{A}\ddot{A}]}V_{[(air)-M]}}\ddot{V}_{[E-(air)]}$ RATE OF CHANGE OF ACCELERATION OF THE WIND INTRODUCES A FORCED ERROR.
$A_{z(cm)} = -S_{[c(CD)][\ddddot{A}\dddot{A}]}(DC)_{(ind)(tc)} - S_{[c(CD)][\dddot{A}\ddot{A}]}\int(DC)_{(ind)(tc)}\,dt$ $-S_{[c(CD)][\ddot{A}\dot{A}]}\iint(DC)_{(ind)(tc)}\,dt\,dt\ S_{[c(CD)][\dot{A}A]}\iiint(DC)_{(ind)(tc)}\,dt\,dt\,dt$	$(C)\overset{\cdot\cdot\cdot\cdot\cdot}{A}_{(Vert)(tc)} + \left\{\frac{g_{1R}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}V_{[(air)-M]}} + \frac{S_{[c(CD)][\dddot{A}\ddot{A}]}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}}\right\}(C)\ddddot{A}_{(Vert)(tc)} + \left\{W^2_{NE} + \frac{S_{[c(CD)][\ddot{A}\dot{A}]}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}}\right\}(C)\dddot{A}_{(Vert)(tc)} + \left\{W^2_{NE}\frac{S_{[c(CD)][\dddot{A}\ddot{A}]}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}} + \frac{S_{[c(CD)][\dot{A}A]}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}}\right\}(C)\ddot{A}_{(Vert)(tc)} + W^2_{NE}\frac{S_{[c(CD)][\ddot{A}\dot{A}]}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}}(C)\dot{A}_{(Vert)(tc)} + W^2_{NE}\frac{S_{[c(CD)][\dot{A}A]}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}}(C)A_{(Vert)(tc)} = \frac{W^2_{NE}}{S_{[c(CD)][\ddddot{A}\dddot{A}]}V_{[(air)-M]}}\dddot{V}_{[E-(air)]}$ THE RATE OF RATE OF CHANGE OF WIND ACCELERATION PRODUCES A FORCED ERROR.

4. Range Indication Mechanization Summary

Examining the control equation obtained in Chapter IV, it is seen that it is possible to eliminate forcing function terms of orders up to and including wind acceleration terms. In a stable system, higher order terms cannot be removed. A general performance function including all mechanization equations that eliminate acceleration and lower order terms can be written

$$(PF)_{(mech)(AA)(long)} = \frac{S_{(n-1)T}\, P + S_{(n)T}}{P^n + S_{(1)T} P^{n-1} + \cdots + S_{(n-2)T} P^2} \qquad (V\text{-}19)$$

If more terms are included in the denominator, velocity and position errors result; if more terms are included in the numerator, control is lost over low order terms, while higher order terms are eliminated. If terms are omitted, instability results.

A block diagram showing a method of mechanizing the performance function eq (V-19) follows, fig (V-5).

For both track control and range indication systems, the response for sinusoidal forcing functions of 84.4 minute period appears to be the same regardless of the choice of damping or order of the equation in the mechanization system.

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$$(PF)_{(mech)(AA)(long)} = \frac{\dfrac{S_{(n-1)T}}{p^{(n-1)}} + \dfrac{S_{(n)T}}{p^{n}}}{1 + \dfrac{S_{(1)T}}{p} + \dfrac{S_{(2)T}}{p^{2}} + \cdots + \dfrac{S_{(n-2)r}}{p^{(n-2)}}} = \frac{S_{(n-1)Tp} + S_{(n)T}}{p^{n} + S_{(1)T}p^{n-1} + \cdots + S_{(n-2)T}p^{2}}$$

TO OBTAIN THE SENSITIVITIES OF THE EQUATION WRITTEN ABOVE, EQUATE COEFFICIENTS WITH PROPER TERM OF BLOCK DIAGRAM EXAMPLE: (for quadratic)

$$A_{r(ind)} = (S_g)\left[\frac{S_2}{p}\left(\frac{S_2}{p}+1\right)(DC)_{(ind)r}\right]$$

$$\text{so } S_{(n)T} \equiv S_2 S_1 \text{ and } S_{(n-1)T} \equiv S_2$$

IMPLEMENTATION OF GENERAL

TION OF GENERAL LONGITUDINAL MECHANIZATION EQUATION

FIG. V-5

APPENDIX A

DISCUSSION OF SIMPLIFYING ASSUMPTIONS

Several simplifying assumptions were made in reaching the mechanization equations and performance equations discussed and analyzed in this thesis. These will now be examined.

1. Omission of the Geodesic Acceleration Term

The geodesic acceleration arises from the fact that navigation is performed on the surface of the earth, where the gravity equipotential forms the geoid. This geoid may be considered to be an ellipsoid with roughness. The deflection of the vertical due to this roughness never exceeds seventy seconds of arc, and (especially over land masses) is usually far smaller.

While the missile travels over this ellipsoid, computation assumes that the surface is a sphere. The shortest distance between two points on this sphere is a great circle. The gravity verticals fall in this great circle plane. The great circle plane transfers to the surface of the ellipsoid as a plane section, but the gravity verticals no longer lie in this plane, but in an arc intersecting it at the points of departure and destination. The missile, traveling along this curved path, experiences a horizontal acceleration. The radius of curvature of this path, which is a minimum at 45 degrees north or south latitude, is never less than one million miles. The geodesic acceleration is therefore of negligible magnitude for missiles with velocities of a few thousand miles an hour.

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Two figures (Fig A-1 and 2) prepared by Dr. W. Wrigley of the M.I.T. Instrumentation Laboratory, illustrate this geodesic acceleration.

2. Compensation for Coriolis Acceleration

The acceleration of the Coriolis arises from the fact that the missile is moving in earth space, which possesses an angular velocity with respect to inertial space. This is demonstrated in the accompanying figure (Fig A-3). Compensation is independent of the heading of the missile, depending only on a knowledge of the earth's angular velocity with respect to inertial space, which is accurately known, on latitude and on ground speed. An indication of latitude and of ground speed are available within the missile, the accuracy of which are dependent on the success of the control system. Errors resulting from inaccuracies in these measurements will be negligible in any practicable system.

A certain amount of cross-coupling of the track control and range systems will occur through the Coriolis Computer, but it will remain small if the component of missile velocity perpendicular to the track is not large.

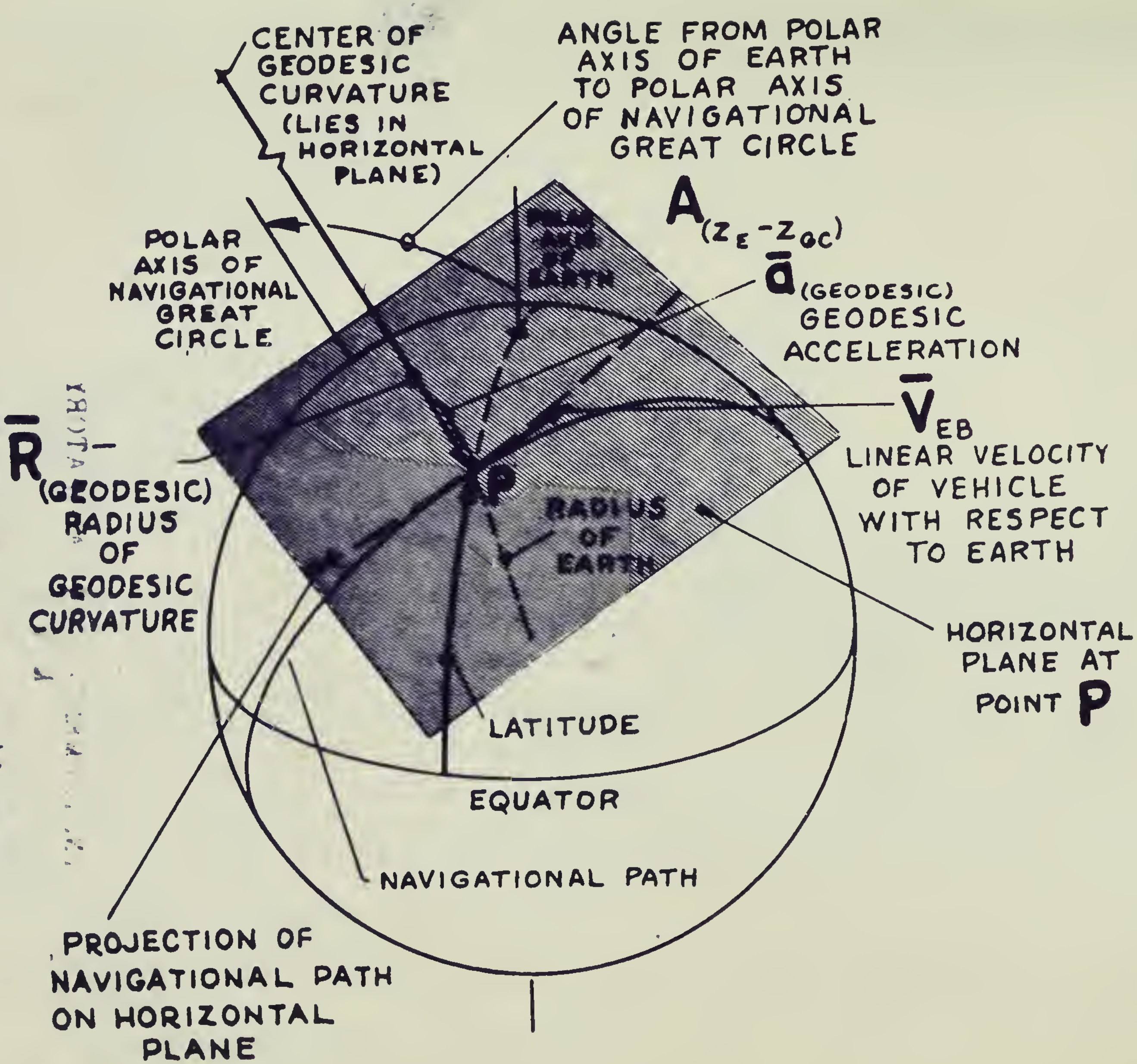
3. Small Angle Pendulum Assumption

For simplicity of calculation, a linearization assumption has been made in the output of the pendulum. The pendulous accelerometer actually solves the equation

$$\frac{a_{[E-(cm)](hor)}}{g_{IR}} = \tan (DC)_{(true)}$$

The control equations are based on the assumption that

$$\frac{a_{[E-(cm)](hor)}}{g_{IR}} = (DC)_{(true)}$$



GEODESIC ACCELERATION IS THE LINEAR ACCELERATION ASSOCIATED WITH THE HORIZONTAL CURVATURE OF THE NAVIGATIONAL PATH.

GEODESIC (HORIZONTAL CURVATURE) ACCELERATION

$$\bar{a}_{(GEODESIC)} = \frac{\bar{V}_{EB} \times (\bar{V}_{EB} \times \bar{R}_{(GEODESIC)})}{R^2_{(GEODESIC)}}$$

$$a_{(GEODESIC)} = \frac{V_{EB}^2}{R_{(GEODESIC)}}$$

$$R_{(GEODESIC)} \cong \frac{R_E \sin(LAT) \cos A_{(Z_E - Z_{OC})}}{2 \times (ELLIPTICITY)}$$

WHERE

$$ELLIPTICITY = \frac{R_{E(eq)} - R_{E(pol)}}{R_E} = \frac{1}{297}$$

IN PRACTICE, THE MINIMUM RADIUS OF GEODESIC CURVATURE ASSOCIATED WITH THE NAVIGATIONAL PATH IS APPROXIMATELY ONE MILLION MILES

FIG. A-1

BASIC GEOMETRIC FACTORS ASSOCIATED WITH GEODESIC ACCELERATION

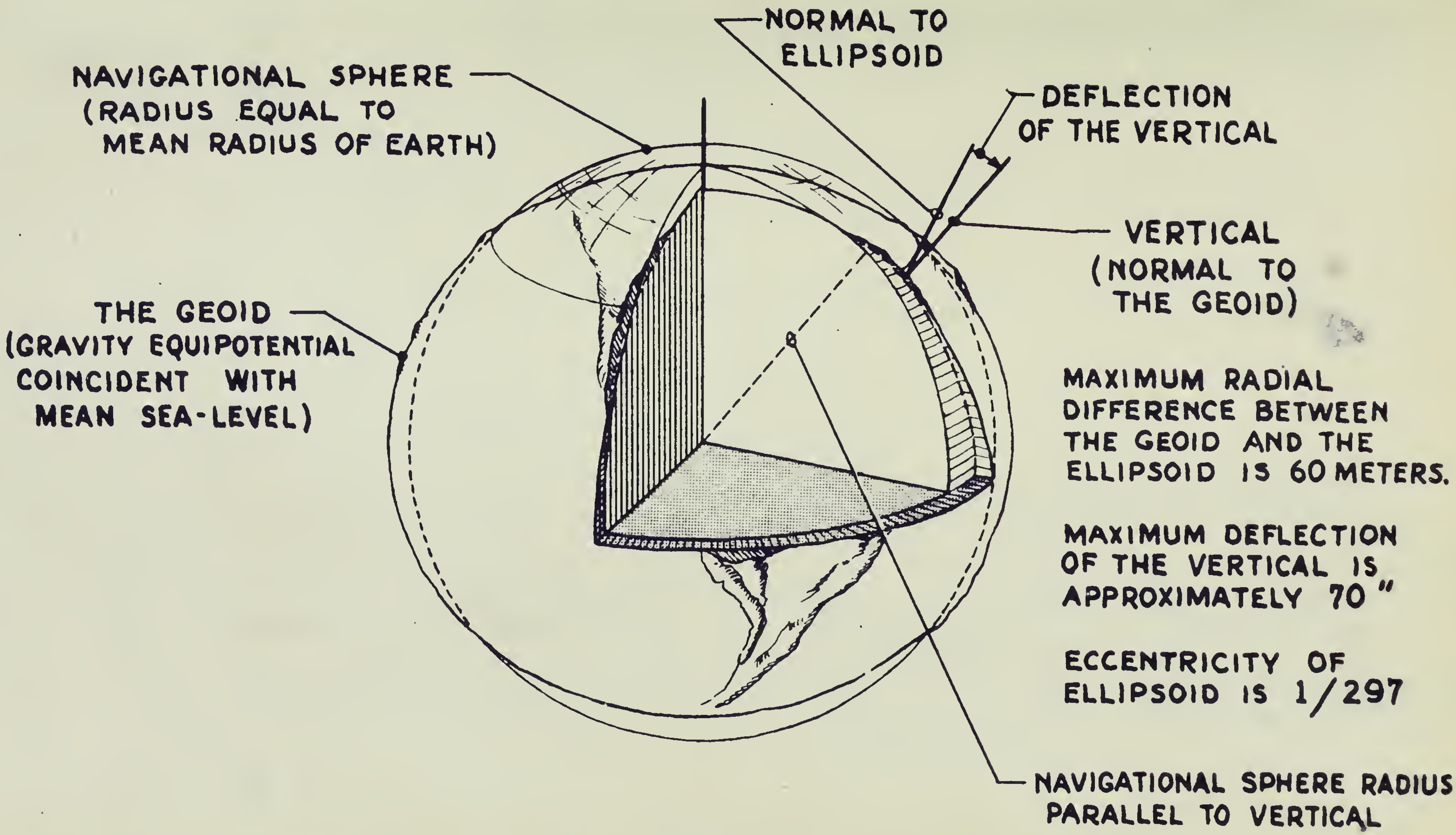


FIG. A-2.

RELATIONSHIPS BETWEEN THE GEOID, REPRESENTATIVE ELLIPSOID, AND THE NAVIGATIONAL SPHERE; DEFLECTION OF THE VERTICAL.

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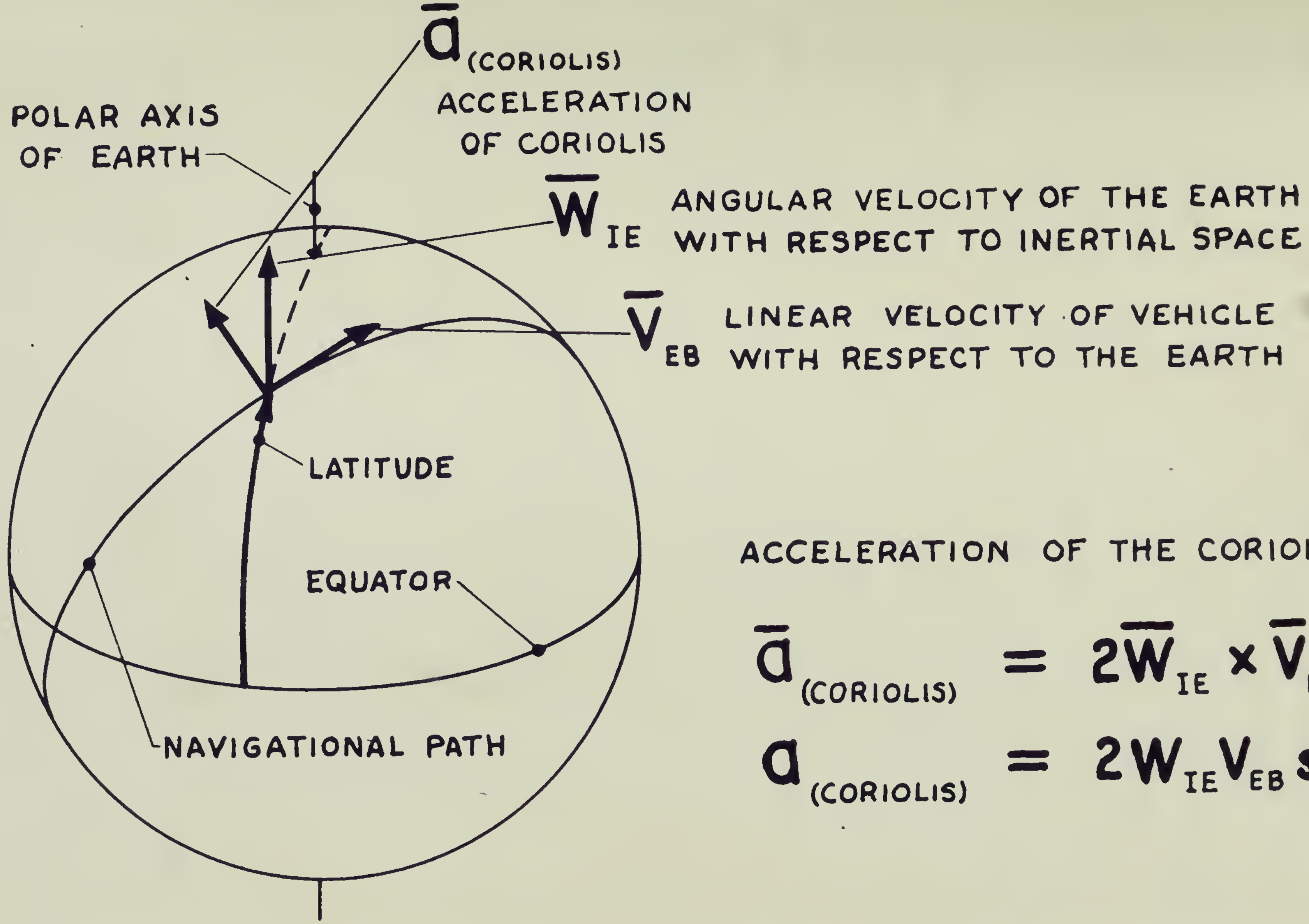

ACCELERATION OF THE CORIOLIS

$$\bar{a}_{(CORIOLIS)} = 2\overline{W}_{IE} \times \overline{V}_{EB}$$

$$a_{(CORIOLIS)} = 2W_{IE}V_{EB}\sin(LAT)$$

FIG. A-3

BASIC GEOMETRIC FACTORS ASSOCIATED WITH THE ACCELERATION OF THE CORIOLIS

The validity of this assumption has been tested, by comparing the output, for the track control system, to a pulse of wind acceleration of 2700 feet/min^2, lasting for one minute, using the transcendental and the simplified equation. The results are shown in the accompanying figure (Fig A-4).

If necessary, a tangent pick-off can be used on the accelerometer instead of the linear pick-off postulated in this discussion.

4. Neglect of Vertical Acceleration

If the missile does not maintain absolutely constant altitude (or if the magnitude of the gravity force varies), the pendulum angle will not be a function of horizontal acceleration alone. This effect, together with the effect of an aerodynamic lag and a pendulum lag, is considered in the following treatment of the range indication problem. There are changes in some of the geometric equations, as indicated:

Aerodynamic lag term:

$$\frac{a_{[E-(cm)](hor)}}{\dot{V}_{[(air)-E]}} = \frac{1}{(CT)_{(aero)}\,p+1} \tag{A-1}$$

Accelerometer lag term:

$$\frac{(DC)_{(true)}}{(DC)_{(true)(corr)}} = \frac{1}{(CT)_{(pu)}\,p+1} \tag{A-2}$$

$$\ddot{A}_{r(true)} = \frac{a_{[E-(cm)](hor)}}{R_E} \tag{A-3}$$

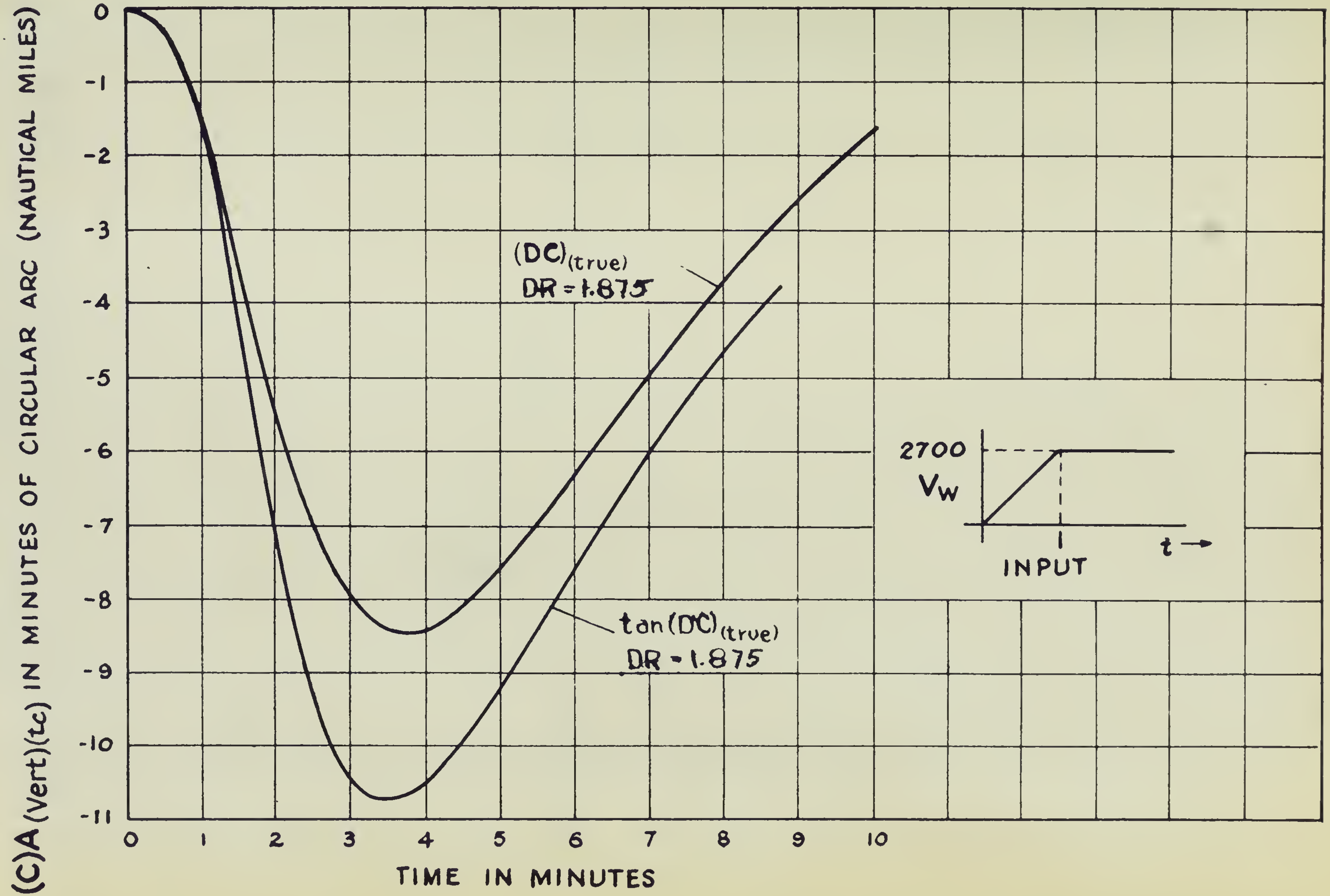

FIG. A-4 EFFECT OF LINEARIZING PENDULUM OUTPUT

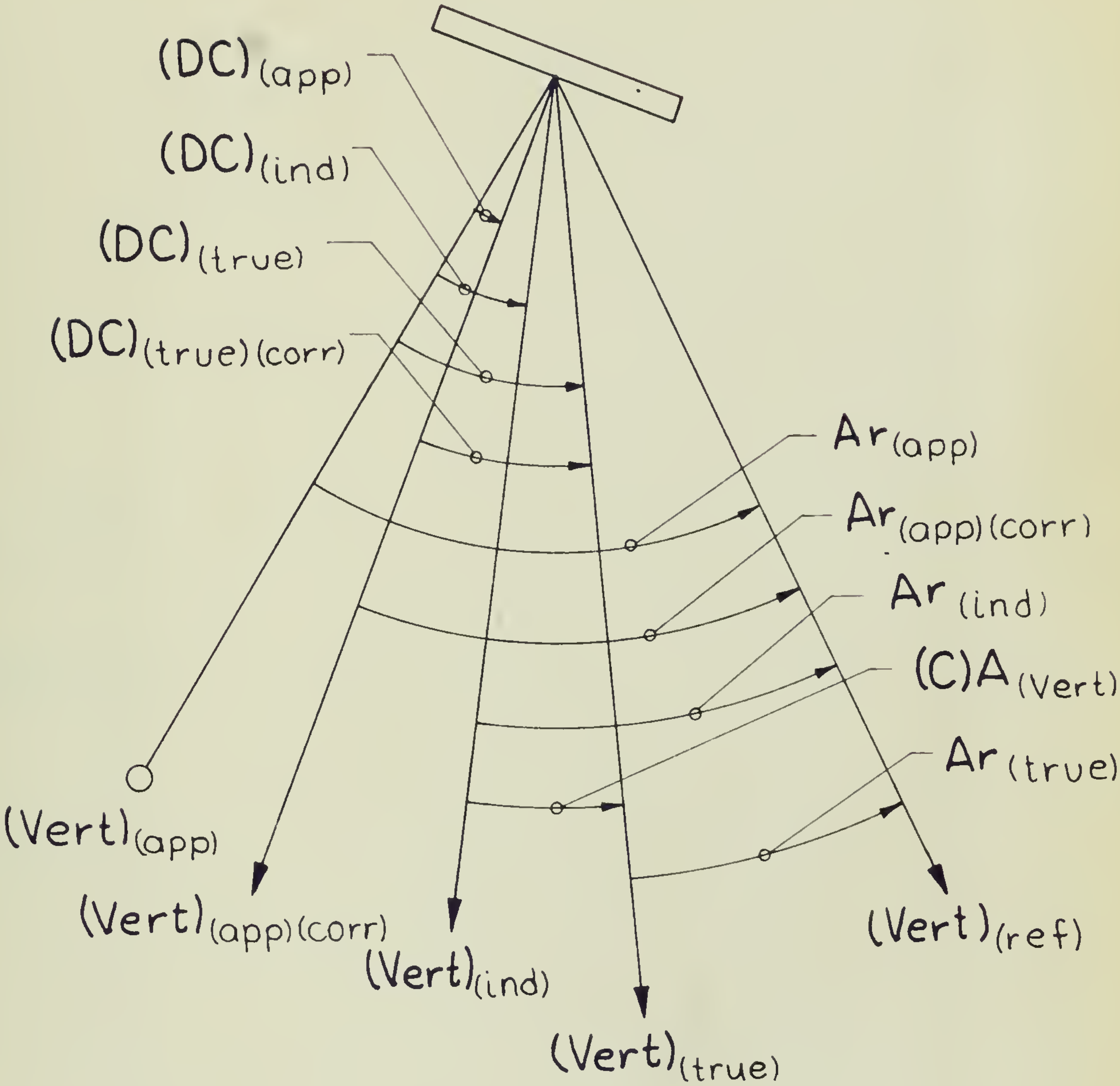

FIG. A-5. INTRODUCTION OF PENDULUM LAG IN RANGE SYSTEM.

$$(DC)_{(true)(corr)} = \frac{a_{[E-(cm)]}}{g_{IR} - a_{[E-(cm)](Vert)}} \tag{A-4}$$

The mechanization chosen was the cubic mechanization equation discussed in Chapter IV.

The control equation for the range indication system, using the concepts outlined above, becomes,

$$\frac{(C)A_{(Vert)}}{\dot{V}_{[E-(air)]}} = \frac{\frac{(g_{IR} - a_{[E-(cm)](Vert)})}{R_E}(CT)_{(pu)}\, p^2 + \left[S_{[c(cm)](\ddot{A}A)} - g_{IR} + a_{[E-(cm)](Vert)} - \frac{S_{[c(cm)][\dot{A}A]}(CT)_{(pu)}\, g_{IR}}{R_E} + \frac{S_{[c(cm)][\dot{A}A]}\, a_{[E-(cm)](Vert)}(CT)_{(pu)}}{R_E}\right] p + S_{[c(cm)][\dddot{A}A]} - \frac{S_{[c(cm)][\dot{A}A]}\, g_{IR}}{R_E} + \frac{S_{[c(cm)][\dot{A}A][E-(cm)](Vert)}}{R_E}}{[p^3 + S_{[c(cm)][\dot{A}A]}\, p^2 + S_{[c(cm)][\ddot{A}A]}\, p + S_{[c(cm)][\ddot{A}A]}][g_{IR} - a_{[E-(cm)](Vert)}][(CT)_{(pu)}\, p+1][(CT)_{(aero)}\, p+1]} \tag{A-5}$$

If the sensitivities are chosen so that the cubic in the denominator becomes the product of a quadratic with (DR) = 0.715 and a period of forty minutes, and a first order term with (CT) = 14 min, eq (A-5) becomes

$$\frac{(C)A_{(Vert)}}{\dot{V}_{[E-(cm)]}} = \frac{\frac{a_{[E-(cm)](Vert)}}{g_{IR}} + \left[\frac{1}{4W_{nE}}\left(6 + \frac{a_{[E-(cm)](Vert)}}{g_{IR}}\right) - (CT)_{(pu)}\frac{a_{[E-(cm)](Vert)}}{g_{IR}}\right] p - \frac{(CT)_{(pu)}}{4W_{nE}}\left(1 - \frac{a_{[E-(cm)](Vert)}}{g_{IR}}\right) p^2}{(g_{IR} - a_{[E-(cm)](Vert)})\left[\frac{1}{4W_{nE}^3}p^3 + \frac{4}{W_{nE}^2}p^2 + \frac{7}{4W_{nE}}p + 1\right][(CT)_{(aevo)}p+1][(CT)_{(pu)}p+1]} \tag{A-6}$$

The effect of vertical acceleration becomes very clear. If the missile momentarily loses lift and "drops", the denominator goes to zero. The numerator, meanwhile, acquires a term which gives a forced error from wind acceleration. This error will change sign with change in the direction of vertical acceleration, but the effect of the denominator will keep the opposing effects from balancing. The damping in the denominator cubic, however, will return the missile to the track after the vertical accelerations have passed.

Even with a downward acceleration greater than the effect of gravity, the system retains stability.

Reasonable values for $(CT)_{(aero)}$ and $(CT)_{(pu)}$ will vary from a fraction of a second to a few seconds. These will, therefore, have little effect on the system.

5. Assumption of Perfect Aerodynamic Response

Paragraph 4 indicates that neglect of aerodynamic response terms can have little effect on the performance of the longitudinal system. In the track control system, however, with a fourth order equation or higher, the system appears stable if a positive pendulum angle causes either a left or a right rudder deflection. It seems, therefore, that even a small aerodynamic lag might cause the system to become unstable. The problem was therefore recalculated for one choice of (DR), using, instead of perfect aerodynamic response, a pair of equations. The first gives the desired heading of the missile as a function of pendulum angle; the second relates this desired heading to the actual heading of the missile through an aerodynamic lag.

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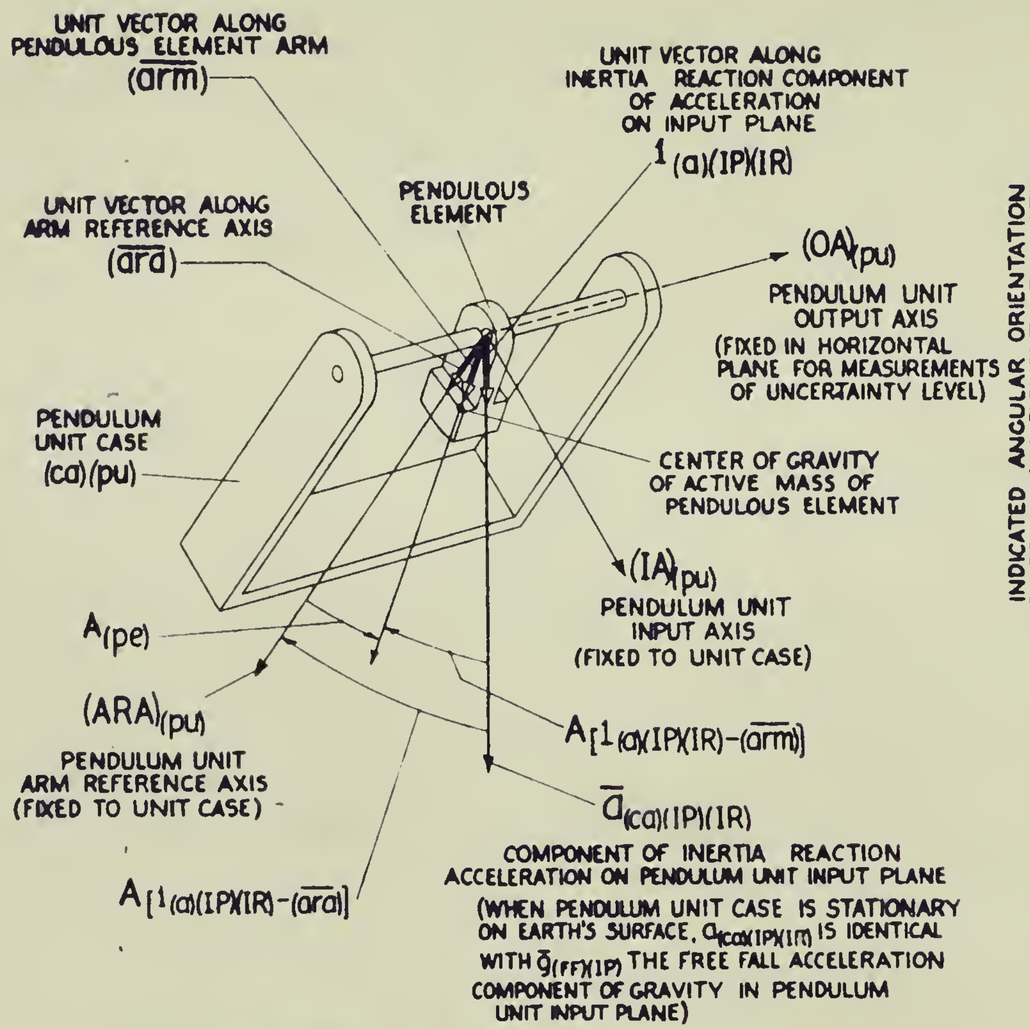
$$A_{Z(cm)(ind)} = S_{[c(CD)][\ddot{A}\ddot{A}]} (DC)_{(ind)(tc)} - S_{[c(CD)][\dot{A}\ddot{A}]} \int (DC)_{(ind)(tc)(dt)} - S_{[c(CD)][A\ddot{A}]} \iint (DC)_{(ind)(tc)} dt\, dt \qquad \text{(A-7)}$$

$$A_{Z(cm)} = \frac{A_{Z(cm)(ind)}}{(CT)_{(aero)}P+1} \qquad \text{(A-8)}$$

The variation of the response of this system from the idealized system discussed in Chapters III and IV could not be detected in the plots drawn by the Differential Analyzer.

6. Assumption of Perfect Gyros and Accelerometers

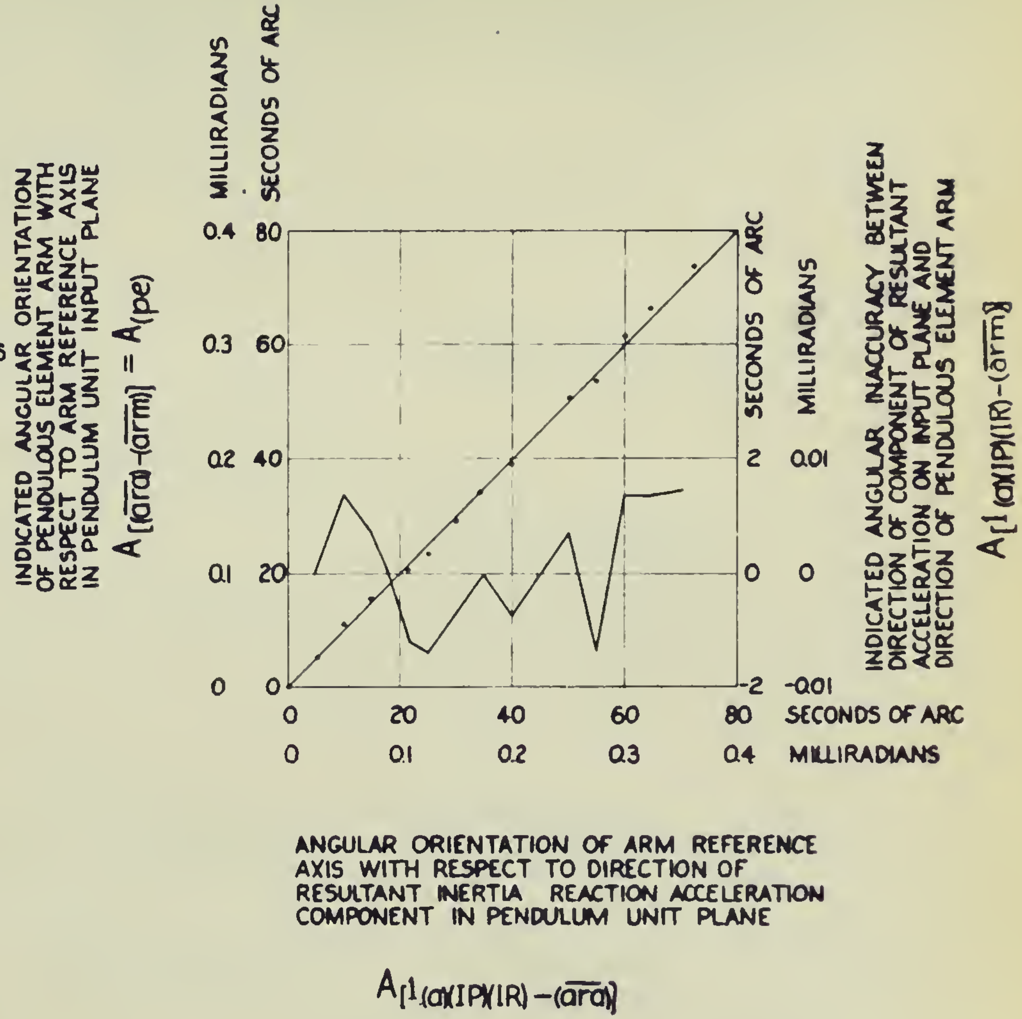
Since this system is an angle measuring system which carries as a reference an indication of the point of departure dependent on the accuracy of the gyros, any drift in the gyros which define the vertical plane will give an equal angular error in range and deflection. The third gyro,



FOR IDEAL PENDULUM UNIT PERFORMANCE

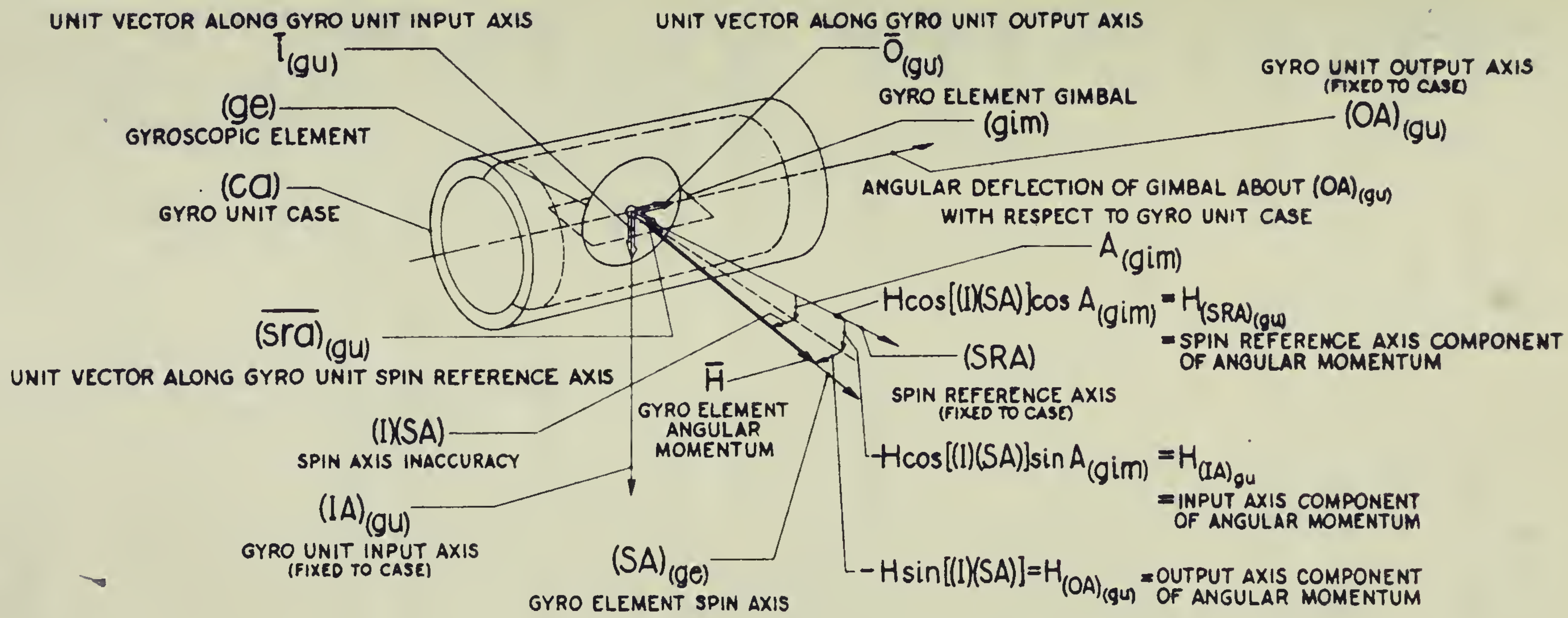
$$A_{[1_{(a)(IP)(IR)}-(\overline{arm})]} = A_{[1_{(a)(IP)(IR)}-(\overline{ara})]} + A_{(pe)} = 0$$

DIAGRAM OF
MEASURED UNCERTAINTY LEVEL
OF SINGLE-DEGREE-OF-FREEDOM
LIMITED ANGLE PENDULUM UNIT



GRAPH OF
MEASURED UNCERTAINTY LEVEL
OF SINGLE-DEGREE-OF-FREEDOM
LIMITED ANGLE PENDULUM UNIT

FIG. A-6.



IN COMPONENT FORM

$$H = \bar{o}_{(gu)} H_{(OA)_{(gu)}} + (sra)_{(gu)} H_{(SRA)_{(gu)}} + \bar{i}_{(gu)} H_{(IA)_{(gu)}}$$
$$= \bar{o}_{(gu)} H \sin[(I)SA] + \overline{(sra)}_{(gu)} H\cos[I(SA)]\cos A_{(gim)} + \bar{i}_{(gu)} H\cos[(I)(SA)]\sin A_{(gim)}$$

$$\bar{W}_{[I(gim)]} = \bar{o}_{(gu)} W_{[I(gim)](OA)_{(gu)}} + \overline{(sra)}_{(gu)} W_{[I(gim)](SRA)_{(gu)}} + \bar{i}_{(gu)} W_{[I(gim)](IA)_{(gu)}}$$

WHERE $\bar{W}_{[I(gim)]}$ IS THE ANGULAR VELOCITY OF THE GIMBAL WITH RESPECT TO INERTIAL SPACE. THE BASIC PERFORMANCE EQUATION FOR A SINGLE-DEGREE-OF-FREEDOM GYROSCOPIC ELEMENT IS

$$\bar{M}_{(ge)(out)} = \bar{H} \times \bar{W}_{[I(gim)]} = \begin{vmatrix} \bar{o}_{(gu)} & \overline{(sra)}_{(gu)} & \bar{i}_{(gu)} \\ H_{(OA)_{(gu)}} & H_{(SRA)_{(gu)}} & H_{(IA)_{(gu)}} \\ W_{[I(gim)](OA)_{(gu)}} & W_{[I(gim)](SRA)(gu)} & W_{[I(gim)](IA)_{(gu)}} \end{vmatrix}$$

FIG. A-7.
COORDINATE DIAGRAM AND STATIC PERFORMANCE CHARACTERISTICS FOR SINGLE-DEGREE-OF-FREEDOM GYRO UNIT WITH NON-IDEAL GEOMETRY.

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which maintains the orientation of the track plane, will introduce a non-linear error. The gyros have been assumed perfect primarily for simplification of the problem. If gyros of sufficient accuracy are found to be impracticable, the necessary inertial reference can be found from observation of the fixed stars.

Permanent errors in the zero indication of the pendulums also introduce an error on the earth's surface equal in angle to the pendulum error.

Single degree of freedom gyros and pendulous accelerometers of types which may prove satisfactory are being developed in the Instrumentation Laboratory at M.I.T. These units are shown diagrammatically in Figs (A-6) and (A-7).

7. Omission of Constants of Integration

Wherever possible, throughout this thesis, constants of integration have been assumed to be zero. For example, in the range indication problem, the missile has been assumed to have no airspeed. This does not change the time response or the size of the range or track error resulting from wind disturbances, but will have a large effect on the distance travelled while the system is recovering from a large disturbance.

At all times, the system has been assumed to be in equilibrium when hit with disturbing wind accelerations in the time solutions of Chapters III and IV.

APPENDIX B

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APPENDIX C

GLOSSARY

The system of notation adopted for use in this thesis has been selected because it fulfills the following desirable objectives:

1. It is easily learned
2. It is adaptable to a wide range of situations
3. It is built up almost exclusively of characters found on the keyboard of a standard American typewriter
4. Any one of the compound symbols of the system is readily interpreted without recourse to an extensive glossary

A few simple examples suffice to explain the operation of the notation system. These examples are shown in table C-3. The short table of key symbols given at the end of this explanation will then furnish sufficient information to enable any compund symbol to be correctly interpreted, and to provide the necessary tools for the generation of new symbols.

A representative list of the primary symbols is given in the following table.

a	Acceleration
A	Angle
(CT)	Characteristic time
[(CT)PR]	Characteristic time - period ratio
(C)	Correction
(DR)	Dampting ratio
(Dir)	Direction
(DC)	Dynamic Correction
(FR)	Frequency ratio
g	Gravity
(PF)	Performance function
R	Radius
S	Sensitivity
(Sg)	Signal
t	Time
V	Velocity
(Vert)	Vertical direction
W	Frequency

PRIMARY SYMBOLS

TABLE C-1

(aero)	Aerodynamic	p	Performance operator (d/dt)
(air)	Air mass	(pend)	Pendulum
(app)	Apparent value	(pu)	Pendulous unit
(aux)	Auxiliary	r	Range
(CD)	Control direction	(ref)	Reference
(cm)	Controlled member	(res)	Resultant
(dep)	Departure	t	Time
(dest)	Destination	(tc)	Track Control; in track control plane
E	Earth	(true)	True value
(hor)	Horizontal	(Vert)	Vertical
(ind)	Indicated value	X	Axis through nose and c.g. of missile
IR	Inertial reaction	Y	Axis through right wing and c.g. of missile
(kin)	Kinematical	Z	Axis through c.g. perpendicular to XY plane, directed downward
(long)	Longitudinal plane; in range direction		
M	Missile		
NE	Earth natural		

MODIFYING SYMBOLS (SUBSCRIPTS)

TABLE C-2

Symbol	Definition
$V_{[E - (cm)]}$	Velocity of the controlled member with respect to the earth
$S_{[C(CD)][A\dot{A}]}$	Sensitivity for correction of the control direction with angle input and rate of change of angle output
$(C)\ A_{(Vert)(tc)}$	Correction to the angle of the vertical in the track control plane
W_{NE}	Natural earth frequency equal to $\frac{g_{IR}}{R_E}$
$A_{Z(cm)}$	Angle about the Z axis of the controlled member
$(DC)_{(ind)(tc)}$	Indicated dynamic correction of track control system
$(DC)_{(ind)(long)}$	Indicated dynamic correction of longitudinal system (i.e. range indication system)
$A_{r(ind)}$	Indicated range angle

Typical Examples Showing How Symbols are Compounded from the Elementary Forms

TABLE C-3

Thesis
E565
Erb
43006
A theoretical study of automatic inertial navigation.